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Eigensensitivity Analysis of Composite Laminates:

Effect of Microstructure

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The purpose of this research is to develop new methods to determine how the frequency of laminates depends upon the microstructure of each individual layer. Toward this end, modern methods of sensititivity analysis are employed to develop new closed-form representations for the frequency response of laminates, where the governing eigenvalue equation is represented either in distributed parameter or discrete form. Solutions are obtained for both damped and undamped structures in terms of the aggregate micromechanical properties (stiffnesses and damping coefficients). Thus the solutions are valid for all macromechanical models of the microstructure.

One new method for determining the effective stiffness and mass properties of the laminate - the Load Correction Method - is developed in detail and some of the computational issues associated with this method are discussed.

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SUMMARY

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ACKNOWLEDGMENT

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1. INTRODUCTION

Overview

The response of multi-layered laminates can be obtained from classical laminate theory [1]. Once the mechanical properties of individual unidirectional laminae are known, the specific eigenvalue equation governing the frequency response of the laminate can be expressed in either discrete or distributed parameter form. According to classical laminate theory, the frequency response of symmetric composite laminates depends upon the stiffness properties (D_{11}, D_{12}, D_{22}, D_{66}, D_{16}, and D_{26}). Likewise, damping can be accounted for by appropriately introducing damping parameters C_{ij}. In turn, the stiffness parameters D_{ij} and damping parameters are known functions of the individual layer stiffness properties E_1, E_2, v_{12}, G_{12} and the layer geometry and orientation. Similarly, C_{ij} can be determined from the damping characteristics of each layer and the partiuclar viscoelastic model that is adopted.

There are numerous papers that develop the laminae stiffness properties in terms of the micromechanical properties E_m, E_f, v_m, v_f and the volume fraction v of, say, fibers. Virturally all studies adopt the law of mixtures[1] to obtain the principal Young's modulus E_1 and Poisson's ratio v_{12} in terms of the aforementioned micromechanical properties. There are some other methods that employ elasticity and its theorems on minimum potential and complementary energy to obtain bounds on E_1 and v_{12} [2-4]. Fortunately, these upper and lower bounds are fairly close to each other. We have found by direct calculation that the frequency response of angle-ply laminates is relatively insensitive to the

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various assumed values of E_1 and v_{12} between their upper and lower bounds.

The stiffnesses E_2 and G_{12} can vary greatly depending upon the assumed modelling assumptions. Likewise, elasticity approaches show that there are wide percentage variations between their permissible maximum and minimum values. Howerver, for most fiber-reinforced laminae, these stiffnesses are small compared to E_1. As a result, the frequency response of angle-ply laminates is relatively independent of the values E_2 and G_{12}, provided these values are selected within their elasticity defined limits. Consequently, insofar as the objective to determine the frequency response of laminated composites in terms of its micromechanical elastic properties is concerned, it appears that the result is relatively independent of the micromechanical model.

Viscoelastic modelling, however, is a different matter. In contrast to its elastic counterpart, there are very few published papers on this subject. In one such paper [5], some viscoelastic properties are dtermined both experimentally and analytically.

A second major objective of this research was to develop new models which retain the local discontinuities of multiphase materials and are capable of characterizing composite fiber-reinforced laminates. The approach was to adapt a methodology, called the load correction method (LCM) [6] for the static analysis of repetitive lattice structures having local discontinuities [7] to the dynamic analysis of composites.

Report Organization

Chapters 2-5 are deveoted to explicity obtaining the solution for the frequency of laminated symmetric plates in terms of the laminates stiffness and damping properties. Chapter's 6 and 7 report on the progress in developing the LCM approach.

Specifically, Chapter 2 is based upon a paper in which the distributed parameter form of the eigenvalue equation for laminates and sensitivity analysis is used to develop an explicity solution for the frequency response of elastic laminates subjected to various boundary conditions. This paper was presented at OPTI '89, an international conference held in Southampton, UK.

In Chapter 3, the general sensitivity analysis method presented by Reiss [7], is specialized to discrete self-adjoint eigenvalue problems. This material was presented in Valparaiso, Chile, 1991 at the 2nd Pan American Congress on Applied Mechanics.

Chapter 4 extends the PI's earlier results to include viscous damping. Sensitivity analysis is again used for both distributed and discrete models. Specific accurate approximations for the complex eigenvalues are given in terms of the stiffness, mass and damping matrices. This chapter was presented at the ASME Design Automation Conference in Chicago, Septermber, 1990.

Chapter 5 employs the methods of Chapter 3 to obtain still new solutions for the frequency response of laminated plates. This solution is slightly more accurate than the one obtained in Chapter 2. Moreover, unlike the method in Chapter 2, this approach is rigorous and can be used to quantify the error in the approximation. These results, originally intended for presentation

at the cancelled IMAC meeting, were presented by invitation at the Florence Modal Analysis Conference, Florence, Italy, in September, 1991.

Chapter 6 summarizes the development of the load correction method to date. Some of the computational aspects of this method are presented a Chapter 7, a paper also presented at PACAM II.

Chapter 8 summarizes some of the conclusions of this report and makes some specific recommendations for further study.

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2. The Fundamental Frequency of Symmetric Laminates Determined by Eigensensitivity Analysis

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ABSTRACT

A new closed-form approximate solution for the fundamental frequency of symmetric rectangular laminates is presented for all classical combinations of clamped and/or hinged edge support conditions. The derivation consists of an expansion of the fundamental frequency in a Maclaurin series in the coupling stiffnesses D_{16} and D_{26} and then truncating after quadratic terms. The required sensitivity derivatives are calculated using a method developed by Reiss [1]. Calculated results obtained from the new approximate formula are generally within 2% of the frequency obtained from conventional numerical methods, but require but a fraction of their computational effort.

INTRODUCTION

An accurate method to determine the flexural fundamental frequency of symmetric rectangular laminates is necessary for modern laminated composite design. Closed-form exact solutions are available only for simply-supported orthotropic laminates [2]. Closed-form approximate, but very accurate, solutions for orthotropic plates, none of whose edges are free, have been obtained by Hearmon [3] using a one-term Rayleigh-Ritz procedure.

The literature also contains a few closed-form approximations for anisotropic laminates. Bert considered general symmetric rectangular laminates whose edges are either simply-supported [4] or clamped [5]. He assumed a linear relationship between eigenvalue and flexural stiffnesses, and determined the unknown constants from numerical calculations. A different approach was taken by Reiss and co-workers. Reiss, Ramachandran and Qian [6], using exact sensitivity derivatives, obtained an approximate solution for the natural frequencies

of a four-layer symmetric simply-supported angle-ply laminate.
The frequencies obtained were presented in the form of a trun-
cated Fourier series in the ply angle. Reiss and Qian [7]
then generalized those results to include all simply-supported
symmetric laminates.

The purpose of this paper is to develop one simple formu-
la to determine, approximately, the fundamental frequency of
symmetric laminates subject to all combinations of hinged and
clamped edge conditions.

PROBLEM STATEMENT

The eigenvalue equation for a freely vibrating symmetric lami-
nate is

$$D_{11} \, w,_{xxxx} + D_{22} \, w,_{yyyy} + 2(D_{12} + 2D_{66}) \, w,_{xxyy}$$

$$+ \, 4 \, D_{16} \, w,_{xxxy} + 4 \, D_{26} \, w,_{xyyy} = \rho \, \omega^2 \, w \qquad (1)$$

where w denotes the mode shape, ω is the corresponding fre-
quency, ρ is the mass density, D_{ij} are the flexural stiff-
nesses [1], x and y are the coordinates in the plane of the
laminate, and indices following a comma denote differentiation
with respect to the indicated argument.

· Each of the edges $x=0$ and $x=a$ is either hinged or clamped.
The appropriate boundary conditions are

hinged: $w - D_{11} \, w,_{xx} + 2 \, D_{16} \, w,_{xy} = 0.$

$$(2)$$

clamped: $w = w,_x \quad 0$

Similarly, each edge $y=0$ and $y=b$ is also either hinged or
clamped. Thus these boundary conditions are

hinged: $w = 2D_{26} \, w,_{xy} + D_{22} \, w,_{yy} = 0$

$$(3)$$

clamped: $w = w,_y = 0$

The objective of this paper is to determine the lowest
frequency ω_1^2 for each of the combinations of boundary condi-
tions (2) and (3) in terms of the material parameters D_{ij} and
plate aspect ratio R:

$$R \equiv a/b \qquad (4)$$

EIGENVALUE DIFFERENTIATION

Before proceeding with the development of the solution to the
stated problem, it is useful to review some general mathemati-

cal formulae that are essential to the current approach.

Consider the class of eigenvalue problems which can be represented in the form

$$T*E(S)T \; w^{mn} = \lambda_{mn} \; M \; w^{mn} \tag{5}$$

to which appropriate mixed boundary conditions must be appended. A detailed discussion of the requirements the operators appearing in Equation (5) and the boundary operators must satisfy can be found in the paper by Reiss and Haug [8]. For our present purposes, it suffices to note that T and T* are L_2 adjoint differential operators, S is a collection of variable parameters, and E and M are positive stiffness and mass operators, respectively. The double indices m and n are included in order to facilitate a solution for the two-dimensional laminate.

The eigenvalue λ_{mn} is a functional of the parameters denoted symbolically by S. And if these parameters change by a small amount δS, λ_{mn} changes by a small amount $\delta\lambda_{mn}$ [1]:

$$\delta\lambda_{mn} = (T \; w^{mn}, \; \delta \; E \; T \; w^{mn}) \tag{6}$$

where the normalized eigenfunctions satisfy

$$(w^{mn}, \; M \; w^{mn}) = 1 \tag{7}$$

In Equations (6) and (7), $(\cdot, \; \cdot)$ denotes the usual L_2 inner product over the domain spanned by the functions involved – in the case, the planar area of the plate. Furthermore, the second variation of the eigenvalue is given by the expression [1]:

$$\delta^2\lambda_{mn} = (T \; w^{mn}, \; \delta^2 E \; T \; w^{mn})$$

$$-2 \; \sum_{i,j \; \neq m,n} \; \frac{(T \; w^{ij}, \delta \; E \; Tw^{mn})^2}{\lambda_{ij} - \lambda_{mn}} \tag{8}$$

In Equation (9), the summation takes place over all pairs of positive integers i,j except i=m and j=n. Equations (6) and (8) are valid provided the eigenvalue λ_{mn} is not repeated.

Similarly, the first variation of an eigenmode w^{mn} associated with a non-repeated eigenvalue λ_{mn} is

$$\delta w^{mn} = - \; \sum_{i,j \; \neq m,n} \; \frac{w^{ij}}{\lambda_{ij} - \lambda_{mn}} \; (T \; w^{ij}, \delta \; E \; T \; w^{mn}) \tag{9}$$

It is clear from Equations (8) and (9), that it is necessary to know the complete set of eigenvalues and eigenfunctions for some parameters S in order to calculate the variations $\delta^2\lambda_{mn}$ and δw^{mn} exactly.

For symmetric laminated plates, we can identify

$$T = [\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\,\partial^2/\partial x\partial y]^T \tag{10}$$

$$T^* = T^T, \quad M = \rho, \quad \lambda_{mn} = \omega_{mn}^2$$

For the specific boundary conditions (2) and (3), we can select the operator E as:

$$E = \begin{bmatrix} D_{11} & D_\kappa & D_{16} \\ D_\kappa & D_{22} & D_{26} \\ D_{16} & D_{26} & 0 \end{bmatrix} \tag{11}$$

where

$$D_\kappa = D_{12} + 2\,D_{66} \tag{12}$$

Therefore the eigenvalues are functions of the five material parameters D_{11}, D_{22}, D_κ, D_{16} and D_{26}.

THE ORTHOTROPIC SOLUTION

For orthotropic laminates, the coupling stiffnesses D_{16} and D_{26} vanish. A complete set of eigenvalues and eigenfunctions can be obtained if D_κ also vanishes. Thus we identify S with D_κ and consider

$$D_{11}\,\hat{w},_{xxxx} + D_{22}\,\hat{w},_{yyyy} = \rho\,\lambda_{mn}\,\hat{w}^{mn} \tag{13}$$

subject to the appropriate boundary conditions (2) and (3) as simplified for an orthotropic plate. The solution, obtained by separating variables, is

$$\hat{w}^{mn} = \rho^{-\frac{1}{2}}\,X_m\left(\frac{x}{a}\right) Y_n\left(\frac{y}{b}\right) \tag{14}$$

$$\rho\,\hat{\lambda}_{mn} = \frac{D_{11}\,\mu_m^4}{a^4} + \frac{D_{22}\,\nu_n^4}{b^4} \tag{15}$$

where X_m and Y_n are the normalized beam shape functions [3] and μ_m and ν_n their corresponding frequencies, i.e.

$$X_m^{IV} - \mu_m^4\,X_m = Y_n^{IV} - \nu_n^4\,Y_n = 0 \tag{16}$$

8

In Equation (16), ( )^{IV} denotes the fourth derivative with respect to the argument of the indicated function.

The solution for the orthotropic laminate can be expressed as a Maclaurin series in D_κ:

$$\bar{\lambda}_{mn} = \hat{\lambda}_{mn} + \frac{\partial \hat{\lambda}_{mn}}{\partial D_\kappa} D_\kappa + \tfrac{1}{2} \frac{\partial^2 \hat{\lambda}_{mn}}{\partial D_\kappa^2} D_\kappa^2 + \ldots \tag{17}$$

where ($\hat{}$) denotes evaluation of the indicated function at $D_\kappa = 0$. The linear term is calculated by observing that

$$\delta E = \frac{\partial \hat{E}}{\partial D_\kappa} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

and then substituting this result together with Equation (14) into Equation (6) to obtain

$$\partial \hat{\lambda}^{mn}/\partial D_\kappa = 2 C_{mm} c_{nn}/\rho a^2 b^2 \tag{19}$$

where

$$C_{ij} = - (X_i', X_j') \qquad c_{ij} = - (Y_i', Y_j') \tag{20}$$

The second derivative is also readily obtained by observing that $\delta^2 E = \partial^2 \hat{E}/\partial D_\kappa^2$ vanishes identically. Now, substitution of Equations (14), (15) and (18) into Equation (8) yields

$$\tfrac{1}{2} \frac{\partial^2 \hat{\lambda}mn}{\partial D_\kappa^2} = -4 \sum_{i,j} \sum_{\neq m,n} \frac{C_{im}^2 c_{nj}^2}{D_{11}(\mu_i^4 - \mu_m^4)b^4 + D_{22}(\nu_j^4 - \nu_n^4)a^4} \tag{21}$$

The orthotropic natural frequencies $\bar{\lambda}_{mn}$ become

$$\rho \bar{\lambda}_{mn} - D_{11} \mu_m^4/a^4 + D_{22} \nu_n^4/b^4 + 2 C_{mm} c_{nn} D_\kappa/a^2 b^2$$

$$- 4 \sum_{i,j} \sum_{\neq m,n} \frac{C_{im}^2 c_{nj}^2 D_\kappa^2}{D_{11}(\mu_i^4 - \mu_m^4)b^4 + D_{22}(\nu_j^4 - \nu_n^4)a^4} + O(D_\kappa^3) \tag{22}$$

Equation (22) is exact to within terms of order D_κ^3. The first three terms on the right-hand side of Equation (22) is precisely the solution obtained by Hearmon [3] using an entirely different approach. The quadratic term in Equation (22) represents a correction to Hearmon's solution. Routine calculations for a wide range of parameters D_{ij} and plate aspect ratios R show that the second order correction is not only negligible for the fundamental frequency of orthotropic plates, but also for higher natural frequencies. The only exceptions

occur for higher frequencies and select values of R, namely those for which the denominator of the quadratic terms becomes small for some pair of indices i and j. This corresponds to the case when Equation (13) admits repeated eigenvalues, that is, when the right-hand side of Equation (15) has the same value for two distinct pairs (m,n).

Using the foregoing approach, the eigenfunctions $\bar{w}^{mn}$ for the orthotropic laminate can be shown to satisfy

$$\rho^{\frac{1}{2}} \bar{w}^{mn} = X_m Y_n - D_\kappa \sum_{i,j} \sum_{\neq m,n} \alpha_{ijmn} X_i Y_j + \ldots \qquad (23)$$

where

$$\alpha_{ijmn} = \frac{2 C_{im} c_{nj}}{D_{11}(\mu_i^4 - \mu_m^4)R^{-2} + D_{22}(\nu_j^4 - \nu_n^4)R^2} \qquad (24)$$

A few observations are in order. For most cases that we calculated, the quantity $\alpha_{ijmn} D_\kappa$ was small compared to unity for all i and j. In such cases, the solution $\hat{w}^{mn}$ is a sufficiently accurate approximation to $\bar{w}^{mn}$. Furthermore, in cases when $\hat{w}^{mn}$ cannot approximate $\bar{w}^{mn}$, the terms $\alpha_{ijmn} D_\kappa$ was found to be comparable to unity only for one set of indices i and j; more importantly, this term contributed a negligible amount to the inner products required for subsequent calculations. Thus, throughout the remainder of this investigation, we will substitute $\hat{w}^{mn}$ for $\bar{w}^{mn}$ in all inner products.

THE ANISOTROPIC SOLUTION

Like the orthotropic solution, the anisotropic solution is developed by expressing it in a Maclaurin series. Here, the parameters S are identified with D_{16} and D_{26}. Thus

$$\lambda_{11} = \bar{\lambda}_{11} + \frac{\partial \bar{\lambda}_{11}}{\partial D_{16}} D_{16} + \frac{\partial \bar{\lambda}_{11}}{\partial D_{26}} D_{26} + \frac{1}{2} \frac{\partial^2 \bar{\lambda}_{11}}{\partial D_{16}^2} D_{16}^2$$

$$\qquad (25)$$

$$+ \frac{1}{2} \frac{\partial^2 \bar{\lambda}_{11}}{\partial D_{26}^2} D_{26}^2 + \frac{\partial^2 \bar{\lambda}_{11}}{\partial D_{16} \partial D_{26}} D_{16} D_{26} + \ldots$$

where the symbol ($\bar{}$) denotes evaluation of the indicated function at the orthotropic solution $D_{16} = D_{26} = 0$.

Since the orthotropic solution is now completely known, albeit approximately, each of the coefficients in the series (25) can be evaluated. These calculations, when substituted

back into Equation (25), lead to the desired closed-form approximation to the fundamental frequency, namely

$$\rho b^4 \lambda_{11} = D_{11} \mu_1^4/R^4 + D_{22} \nu_1^4 + 2 D_\kappa C_{11}c_{11}/R^2$$

$$- 4 R^{-2} \Sigma \Sigma [(H_{i1}g_{1j}+H_{1i}g_{j1})D_{16}+(G_{1i}h_{j1}+G_{i1}h_{1j})R^2 D_{26}]^2 \times$$

$$[D_{11}(\mu_i^4-\mu_1^4)+D_{22}(\nu_j^4-\nu_1^4)R^4 + \qquad (26)$$

$$2D_\kappa R^2(C_{ii}c_{jj} - C_{11}c_{11})]^{-1}$$

where

$$G_{ij} = (X_i', X_j) \qquad\qquad g_{ij} = (Y_i', Y_j)$$

$$\qquad\qquad (27)$$

$$H_{ij} = (X_i'', X_j') \qquad\qquad h_{ij} = (Y_i'', Y_j)$$

Apart from the orthotropic approximations employed in this derivation, Equation (26) is correct to within quartic terms in the coupling stiffnesses.

Inclusion of higher order terms in the Maclaurin series is unlikely to yield any significant improvement in the accuracy of the approximation (26) unless the orthotropic solution is also improved. The only exception is the simply-supported plate for which the beam shape functions provide the exact orthotropic solution. Nevertheless, even for this plate, the addition of higher order terms negates the simplicity of the approximation. Furthermore, as the numerical illustrations in Tables I-III demonstrate, Equation (26) is remarkably accurate.

NUMERICAL EXAMPLES

In the following tables, the validity of Equation (26) is established by comparing the fundamental frequency calculated from it with the fundamental frequency obtained numerically. Since Equation (26) uses the vibrating beam shape functions, the numerical approach selected was the Ritz method using these same shape functions. Thus the fundamental mode is expressed

$$w^{11} = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} X_i(x/a) Y_j(y/b) \qquad (28)$$

It is convenient to introduce the following non-dimensional fundamental frequency.

$$k_1 = \omega_{11} \, b^2 \, \sqrt{\rho/U_1 h^3} \qquad\qquad (29)$$

where U_1 is an invariant material property [2] and h is the common thickness of the plies.

Results are presented for balanced symmetric four-ply plates [$\theta/-\theta/-\theta/\theta$]. Each table includes results for typical boron epoxy (B/E) and high modulus graphite epoxy (G/E) defined by the following data.

	B/E	G/E
E_1/E_2	10	40
E_1/G_{12}	40	80
ν_{12}	.30	.25

The tables have been developed for different boundary conditions, whose nomenclature is S for a simply supported edge and C for a clamped edge. The first letter denotes the support condition on the edge x=0 and the subsequent letters are for edges proceeding counterclockwise around the plate. For example CCCS means clamped on all sides except y=b where it is simply supported. The columns labelled $\bar{k}_1$ are calculated by the Ritz method with N=11 i.e. 121 terms in Equation (28). In most cases, the Ritz method has converged with N=11; in those cases for which convergence is not evident for N=11, comparison with the N=7 and N=9 term solutions suggests that the indicated value is within 1% of the correct frequency.

It should be observed that in all cases presented the approximate solution (26) is within 2% of the Rayleigh-Ritz solution, and often much better than that. Calculations for other boundary conditions and other material data do not alter this conclusion.

ACKNOWLEDGMENT

TABLE I: RESULTS FOR CCCC PLATE

θ°	B/E		G/E	
	$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
			R=1	
0	84.17	84.19	84.28	84.30
15	82.26	82.38	81.59	81.79
30	78.65	79.19	76.62	77.49
45	77.00	77.76	74.46	75.59
			R=5	
0	25.28	25.28	13.64	13.65
30	31.51	31.61	25.74	25.86
60	61.21	61.26	62.95	62.99
90	78.65	78.66	82.54	82.54

TABLE II: RESULTS FOR SCSC PLATE

θ°	B/E		G/E	
	$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
			R=1	
0	80.82	80.83	83.33	83.33
30	74.27	74.40	75.47	75.62
60	57.36	57.96	55.05	55.93
90	45.33	45.36	39.96	39.97
			R=5	
0	11.81	11.81	6.94	6.94
30	15.67	15.87	13.44	13.65
60	27.64	27.73	28.33	28.42
90	34.80	34.81	36.46	36.46

TABLE III: RESULTS FOR CCCS PLATE

θ°	B/E		G/E	
	$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
		R=1		
0	61.84	61.87	59.34	59.36
30	64.70	65.44	61.54	62.69
60	75.01	75.42	74.23	75.27
90	82.10	82.13	83.70	83.72
		R=5		
0	25.16	25.16	13.42	13.42
30	31.19	31.50	25.64	24.76
60	61.19	61.24	62.94	62.99
90	78.64	78.65	82.53	82.54

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3. EIGENVALUES OF SELF-ADJOINT SYSTEMS DETERMINED BY EIGENSENSITIVITY ANALYSIS

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ABSTRACT

A new method is presented to determine approximate closed-form solutions for the eigenvalues of self-adjoint structures with non-uniform stiffness and mass distributions. The distributed parameter eigenvalue equation is cast into an equivalent discrete formulation with an infinite number of degrees of freedom. The resulting stiffness and mass matrices are each decomposed into two matrices, one diagonal and the other with zero diagonal elements. Sensitivity derivatives are used to expand the eigenvalues in a power series of the zero diagonal matrices.

INTRODUCTION

Eigensensitivity methods have been used primarily in optimum design or reanalysis of structures. Recently, these methods have been used to obtain approximate closed-form solutions to specific eigenvalue problems [1]. In this paper, eigensensitivity analysis is used to obtain a simple expression which approximates the eigenvalues for a wide-class of self-adjoint eigenvalue problems.

Consider the eigenvalue problem in abstract form

$$T^* E(S) T u_i = \lambda_i M(S) u_i \tag{1}$$

subject to appropriate mixed boundary conditions. Here, T and T^* are L_2 adjoint differential operators, E is the positive, symmetric stiffness operator, M is another positive symmetric operator and S denotes a collection of parameters upon which both E and M depend. Furthermore λ_i is the eigenvalue with associated eigenfunction u_i.

There are many ways to represent Eq. (1) by an equivalent discrete system. In this paper, the Rayleigh-Ritz method will be used. If $\{\psi_n\}$ is a complete kinematically admissible set of basis functions, and

$$u_i = \sum_n \alpha_{in} \psi_n \tag{2}$$

then stationarity of the Rayleigh quotient requires

$$\mathbf{K} \alpha_i = \lambda_i \mathbf{M} \alpha_i \tag{3}$$

where the components K_{ij} and M_{ij} of the stiffness matrix $\mathbf{K}$ and the matrix $\mathbf{M}$ respectively satisfy

$$K_{ij} = (T\psi_i, ET\psi_j) \tag{4}$$

$$M_{ij} = (\psi_i, M\psi_j)$$

Here, $(\cdot, \cdot)$ denotes the L_2-inner product. Also α_{in} is the n-th component of the infinitely dimensional eigenvector $\hat{\alpha}_i$.

FORMAL SOLUTION

It is desired to solve Eq. (3) for the eigenvalues λ_i. Toward this end it is convenient to introduce parameters S_1 and S_2 and the one parameter family of matrices $\mathbf{K}(S_1)$ and $\mathbf{M}(S_2)$ defined by

$$\hat{\mathbf{K}}(S_1) = \mathbf{K}_D + S_1 \Delta \mathbf{K} \tag{5}$$

$$\hat{\mathbf{M}}(S_2) = \mathbf{M}_D + S_2 \Delta \mathbf{M}$$

In Eq. (5), $\mathbf{K}_D$ and $\mathbf{M}_D$ are diagonal matrices whose components consist, respectively, of the diagonal elements of $\mathbf{K}$ and $\mathbf{M}$; also $\Delta \mathbf{K}$ and $\Delta \mathbf{M}$ have zero diagonal elements, and off diagonal components equal to those of $\mathbf{K}$ and $\mathbf{M}$, respectively. Evidently,

$$\hat{\mathbf{K}}(1) = \mathbf{K} \qquad\qquad \hat{\mathbf{M}}(1) = \mathbf{M} \tag{6}$$

Consider the eigenvalue equation

$$\hat{\alpha}_i(S_1, S_2) = \hat{\lambda}_i(S_1, S_2) \hat{\mathbf{K}}^{-1}(S_1) \hat{\mathbf{M}}^{-1}(S_2) \hat{\alpha}_i(S_1, S_2) \tag{7}$$

If $S_1 = S_2 = 0$, $\hat{\mathbf{K}}^{-1}$ and $\hat{\mathbf{M}}^{-1}$ are diagonal, and therefore Eq. (7) readily admits the solution

$$\hat{\lambda}_i(0,0) = K_{ii} M_{ii}^{-1} \qquad, \qquad \hat{\alpha}_i(0,0) = M_{ii}^{-\frac{1}{2}} e_i \tag{8}$$

where e_i is the unit Cartesian base vector.

Equation (7) is a special case of the eigenvalue equation discussed by Reiss [2] in which an explicit representation for the variations of the eigenvalues and eigenvectors was determined. In the context of Eq. (7), these variations become

$$\delta\hat{\lambda}_i = \hat{a}_i \Delta K \hat{a}_i \, \delta S_1 - \hat{\lambda}_i \hat{a}_i \Delta M \hat{a}_i \, \delta S_2$$

(9)

$$\delta^2\hat{\lambda}_i = -2\sum_{n\neq i} \frac{(\hat{\lambda}_i \hat{a}_n \Delta M \hat{a}_i \delta S_2 - \hat{a}_n \Delta K \hat{a}_i \, \delta S_1)^2}{\hat{\lambda}_n - \hat{\lambda}_i}$$

(10)

$$- 2\delta\hat{\lambda}_i \hat{a}_i \Delta M \hat{a}_i \, \delta S_2$$

provided the eigenvectors a_i are normalized with respect to $\hat{M}$. Furthermore, the variations (9) and (10) satisfy

$$\hat{\lambda}_i(S_1 + \delta S_1, S_2 + \delta S_2) - \hat{\lambda}_i(S_1 S_2) = \delta\hat{\lambda}_i + \frac{1}{2}\delta^2\hat{\lambda}_i + \ldots$$

(11)

Since $\hat{\lambda}_i(1,1) \equiv \lambda_i$, the solution for λ_i may be obtained by substituting Eqs. (9) and (10) into Eq. (11) and evaluating at $S_1 = S_2 = 0$, $\delta S_1 = \delta S_2 = 1$. Thus if only second order terms in δS_i are retained,

$$\lambda_i = K_{ii}M_{ii}^{-1} - M_{ii}^{-2}\sum_{n\neq i} \frac{(M_{ii}\Delta K_{ni} - K_{ii}\Delta M_{ni})^2}{M_{ii}K_{nn} - M_{nn}K_{ii}}$$

(12)

Even higher order variations could be calculated, however, only at the expense of the simplicity of the representation (12).

EXAMPLE

Consider the torsional vibrations of a fixed-free shaft whose radius r varies with the axial dimensions x. The non-dimensional eigenvalue equation is

$$- (r^4(x) u')' = \lambda r^4(x) u$$

where $u(0) = u'(1) = 0$ and $(\ )' = d/dx$. A convenient basis is

$$\psi_n = \sin\left(n - \frac{1}{2}\right)\pi x \ , \quad n = 1, 2, \ldots$$

An approximate solution for λ_i can be obtained from Eqs. (4) and (12) for any r(x). Assuming $r^2 = a - x$, $a \geq 1$, the approximate eigenvalues obtained from Eq. (12) are tabulated in Table 1 for

two different values of a. Note that as $a \to \infty$, the shaft tends to a uniform one and ψ_n approaches the exact eigenfunctions. The first column is the zeroth order solution K_{ii} M_{ii}^{-1} obtained by ignoring the off-diagonal elements of **M** and **K**. The next column represents the second order correction obtained by adding $\delta^2 \lambda_i$ to the first column. Finally, since this tapered shaft admits an exact solution, the last column contains the exact eigenvalues.

Table 1. Approximate eigenvalues for shaft

i	a = 1.5			a = 3		
1	5.417	5.211	5.239	3.407	3.371	3.373
2	24.13	25.24	25.88	23.01	23.17	23.19
3	63.56	63.96	65.55	62.48	62.60	62.68
4	122.8	123.1	124.8	121.7	121.9	121.9
5	201.7	202.1	203.8	200.7	200.8	200.9

Other illustrations, such as the vibration of non-uniform beams and buckling of bars with variable cross-section can be solved with similar ease and accuracy.

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4. EIGENVALUES FOR MODERATELY DAMPED LINEAR SYSTEMS DETERMINED BY EIGENSENSITIVITY ANALYSIS

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ABSTRACT

A new method is presented to determine approximate closed-form solutions for the complex-valued frequencies of moderately damped linearly elastic structures. The approach is equally applicable to finite degree of freedom systems and distributed parameter systems. The damping operator is split into two components, the first of which uncouples the quadratic eigenvalue equation, and the eigenvalues are expressed as a power series in the second component of the damping operator. Specific numerical examples include both finite degree of freedom and distributed parameter systems. It is shown that for moderate damping, that is, when the second component of the damping operator is small, but not negligible, the series solution truncated after quadratic terms provides an excellent approximation to the true eigenvalues.

INTRODUCTION

Eigensensitivity analysis is an exceptionally useful tool for the reanalysis of eigenvalues and eigenvectors. Although it is essentially a perturbation technique, such methods have been used successfully to determine optimum eigenvalue designs as well as approximate closed-form expressions for the eigenvalues of structural systems.

In eigensensitivity techniques, the eigenvalues and eigenvector are assumed to be piecewise differentiable functions of some specified set of parameters, called design parameters. It is the derivatives of the eigenvalues and eigenvectors with respect to the design parameters that are sought. For systems with a finite number of degrees of freedom, Fox and Kapoor (1968) and Plaut and

Huseyin (1973), are among the first to develop explicit closed-form expressions for the design derivatives of self-adjoint and non self-adjoint systems, respectively. Sensitivity derivatives for distributed parameter systems were presented recently by Reiss (1986) by casting the governing self-adjoint differential equation into integral form using Green's function.

In this study, the sensitivity derivatives of the complex eigenvalues associated with viscously damped structures are explicitly determined. Explicit dependence of the eigenvalues and eigenfunctions upon boundary conditions is eliminated by presenting the quadratic eigenvalue equation in integral form. For simplicity, it is assumed that the design variables affect the eigenvalues and eigenfunctions only through the damping operator.

PROBLEM STATEMENT

The closed, bounded and regular domain of an elastic structure is denoted by $\Omega + \partial\Omega$ where the interior of the domain is Ω and its boundary is $\partial\Omega$. Furthermore, $\partial\Omega$ is the union of two regular complementary subsets $\partial\Omega_1$ and $\partial\Omega_2$. A typical point in $\Omega + \partial\Omega$ is denoted by x and a typical time is denoted by t. The design variables, which may depend upon x but not t, are collectively denoted by $S(x)$.

The unforced motion of a linear elastic structure vibrating in a viscous medium (distributed damping) can be represented by the equation

$$T^*ETu + C(S)\dot{u} + M\ddot{u} = 0 \quad (x,t)\in\Omega\times(0,\infty) \qquad (1a)$$

subject to the mixed boundary conditions

$$B u = 0 \quad (x,t)\in \partial\Omega_1\times[0,\infty) \qquad (1b)$$

$B^*ET\ u = 0 \qquad (x,t)\epsilon \quad \partial\Omega_2 \times [0, \infty)$ (1c)

Here, T and T^* are L_2 adjoint differential operators, B and B^* are the corresponding adjoint boundary operators, and the positive, symmetric operators E, M and C are, respectively, the stiffness, mass, and damping operators. The response u depends upon both x and t, and the dot above a symbol indicates its time derivative.

The eigenvalue problem under consideration is obtained by substituting the separable form

$$u(x,t) = U(x)\ e^{\omega t}$$

into Eqs. (1a,b,c). This results in the quadratic eigenvalue equation

$T^*ET\ U_i = -\omega_i C(S)U_i - \omega_i^2 M\ U_i \quad x\ \epsilon\ \Omega$ (2a)

$B\ U_i = 0 \qquad x\ \epsilon\ \partial\Omega_1$ (2b)

$B^*ET\ U_i = 0 \qquad x\ \epsilon\ \partial\Omega_2$ (2c)

In Eq. (2), ω_i is the complex-valued eigenvalue and U_i is the associated eigenfunction. Following Reiss (1986), Eqs. (2) are recast in the simpler single integral equation

$U_i(y) = -\omega_i\ (G(\cdot,y),\ C(S)\ U_i)_\Omega - \omega_i^2(G(\cdot,y),\ MU_i)_\Omega$ (3)

where

$$(a,b)_\Omega = \int_\Omega \bar{a}\ b\ dx$$

and G is the Green's function associated with the operators on the left side of Eqs. (2), and $\bar{a}$ is the complex conjugate of a.

It is clear that the eigenvalues ω_i and eigenfunctions U_i are functionals of the design variables S, and they depend on S only through the damping operator C. If S is changed by an amount δS, then ω_i and U_i will change by $\Delta\omega_i$ and ΔU_i, respectively, where

$$\Delta\omega_i = \omega_i(S + \delta S) - \omega_i(S)$$
$$\Delta U_i = U_i(S + \delta S) - U_i(S)$$
(4)

Assuming, ω_i possesses at least two Gateaux derivatives with respect to S, and U_i at least one Gateaux derivative with respect to S, then Eq. (4) may be expressed

$$\Delta\omega_i = \delta\omega_i + \delta^2\omega_i + O(\delta S^3)$$
$$\Delta U_i = \delta U_i + O(\delta S^2)$$
(5)

where $\delta\omega_i$ and δU_i are linear in δS and $\delta^2\omega_i$ is quadratic in δS. A major objective of this paper is to determine explicit representations for $\delta\omega_i$, $\delta^2\omega_i$ and δU_i in terms of the eigenvalues, eigenfunctions and operators appearing in Eq. (2a).

DESIGN DERIVATIVES

The eigenvalue equation (3) is not self-adjoint and, therefore, the eigenfunctions are not orthogonal in any useful sense. Following Plaut and Huseyin (1973), Eq. (3) can be reduced to standard form by introducing $V_i = -\omega_i U_i$ and the identity operator δ with respect to $(\cdot,\cdot)_\Omega$. Thus Eq. (3) becomes

$$U_i = -\omega_i\ (G,\ C\ U_i + M\ V_i)_\Omega$$
$$V_i = \omega_i\ (\delta,\ U_i)_\Omega$$
(6)

In matrix form, Eq. (6) is

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} = -\omega_i \begin{pmatrix} G\cdot & 0 \\ 0 & \delta\cdot \end{pmatrix} \begin{pmatrix} C & M \\ -1 & 0 \end{pmatrix} \begin{pmatrix} U_i \\ V_i \end{pmatrix}$$
(7)

where the operators $G\cdot$ and $\delta\cdot$ are defined through

$$G\cdot \equiv (G,\ \cdot)_\Omega$$
$$\delta\cdot \equiv (\delta,\ \cdot)_\Omega$$
(8)

In Eqs. (8), G is the same self-adjoint Green's function appearing in Eq. (3).

The first variation of Eq. (7) is now desired. Thus it is assumed that S changes by δS, so that the first variation of Eq. (7) is

$$\delta\begin{pmatrix} U_i \\ V_i \end{pmatrix} + \omega_i \begin{pmatrix} G\cdot & 0 \\ 0 & \delta\cdot \end{pmatrix} \begin{pmatrix} C & M \\ -1 & 0 \end{pmatrix} \delta\begin{pmatrix} U_i \\ V_i \end{pmatrix} = \begin{pmatrix} F_i \\ H_i \end{pmatrix}$$
(9)

where

$$\begin{pmatrix} F_i \\ H_i \end{pmatrix} = -\omega_i^{-1}\delta\omega_i \begin{pmatrix} U_i \\ V_i \end{pmatrix} - \omega_i \begin{pmatrix} (G,\ \delta C U_i)_\Omega \\ 0 \end{pmatrix}$$
(10)

The solution to Eq. (9) can be conveniently obtained by introducing the eigenvalue equation which is adjoint to Eq. (7). Toward this end, define the eigenvalue problem

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = -\omega_i \begin{pmatrix} C & -1 \\ M & 0 \end{pmatrix} \begin{pmatrix} G\cdot & 0 \\ 0 & \delta\cdot \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$
(11)

where $(X_i\ Y_i)^T$ is the adjoint eigenvector. It is straightforward to show that Eqs. (7) and (11) admit the same eigenvalues ω_i, and that the adjoint eigenfunctions are orthogonal in the sense that

$$(\bar{X}_i\ \bar{Y}_i)\cdot\begin{pmatrix} U_j \\ V_j \end{pmatrix} \equiv (\bar{X}_i,\ U_j)_\Omega + (\bar{Y}_i,\ V_j)_\Omega = 0$$

for $i\neq j$. At this point, we are at liberty to arbitrarily select two normalization factors, one for each of the eigenvectors. Consequently, without loss in generality, one set of normalization factors will be selected so that

$$(\bar{X}_i \ \bar{Y}_i) \cdot \begin{pmatrix} U_j \\ V_j \end{pmatrix} \equiv \delta_{ij} \tag{12}$$

where δ_{ij} is the Kronecker delta.

In order to compute the first variation $\delta\omega_i$, we observe that

$$(\bar{X}_i \ \bar{Y}_i) \cdot \begin{pmatrix} F_i \\ H_i \end{pmatrix} \equiv 0. \tag{13}$$

Equation (13) is established by substituting Eq. (9) into the left side of Eq. (13), and changing the order of integration in the resulting double integrals using the symmetry properties of C, M and G. By substituting Eq. (10) into Eq. (13), $\delta\omega_i$ is immediately determined. Thus

$$\delta\omega_i = \omega_i^2 \ (\bar{X}_i, \ (G, \delta CU_i)_\Omega)_\Omega \tag{14}$$

Equation (14) provides an explicit representation for the first variation of the eigenvalue.

The first variation of the eigenfunctions are not so simply obtained. By virtue of Eqs. (10) and (14), the right side of Eq. (9) is explicitly known. It therefore remains to solve the inhomogeneous linear integral equation (9) for $\delta(U_i \ V_i)^T$. Toward this end, let the solution be represented in terms of the eigenfunctions, i.e.

$$\delta \begin{pmatrix} U_i \\ V_i \end{pmatrix} = \sum_n a_{ni} \begin{pmatrix} U_n \\ V_n \end{pmatrix} \tag{15}$$

Further, let $(F_i \ H_i)^T$ also be expanded in terms of these same eigenfunctions. Thus

$$\begin{pmatrix} F_i \\ H_i \end{pmatrix} = \sum_n f_{ni} \begin{pmatrix} U_n \\ V_n \end{pmatrix} \tag{16}$$

Now substituting Eqs. (15) and (16) into Eq. (9), simplifying the result using Eq. (6), and comparing term by term, we find that

$$a_{ij} = \frac{f_{ij} \ \omega_i}{\omega_i - \omega_j} \qquad i \neq j \tag{17}$$

The coefficients f_{ij}, obtained directly from Eqs. (12) and (16), satisfy

$$f_{ni} = (\bar{X}_n \ \bar{Y}_n) \cdot \begin{pmatrix} F_i \\ H_i \end{pmatrix} \tag{18}$$

Now, substitution of Eqs. (10) and (14) into Eq. (18) immediately yields

$$f_{ni} = \begin{cases} -\omega_i \ (\bar{X}_n, (G, \delta CU_i)_\Omega)_\Omega & n \neq i \\ 0 & n = i \end{cases} \tag{19}$$

The first variation δU_i, obtained from Eqs. (15), (17) and (19), becomes

$$\delta U_i = \sum_{n \neq i} \frac{\omega_i \ \omega_n}{\omega_i - \omega_n} \ (\bar{X}_n, \ (G, \delta CU_i)_\Omega)_\Omega \ U_n + a_{11} \ U_i \tag{20}$$

Similarly, it can be shown that the first variation of the adjoint eigenfunction $:X_i$ satisfies

$$\delta X_i = \sum_{n \neq i} \frac{\omega_i \ \omega_n}{\omega_i - \omega_n} \ (\bar{X}_i, (G, \delta CU_n)_\Omega)_\Omega \ X_n - a_{11} \ X_i \tag{21}$$

The constants a_{ii}, as yet, are indeterminate. This is because only one normalization scheme has been involved for the two sets of eigenfunctions. Prescription of the second normalization factor will render a_{ii} determinate.

The second variation of the eigenvalue $\delta^2\omega_i$ can be computed directly from Eqs. (14), (20) and (21). Although some manipulations are required, it is nevertheless straightforward to establish that

$$\delta^2\omega_i = 2 \ \omega_i^3 \ (\bar{X}_i, (G, \delta CU_i)_\Omega)_\Omega^2 + \omega_i^2 \ (\bar{X}_i, (G, \delta^2 CU_i)_\Omega)_\Omega$$
$$+ 2 \ \omega_i^2 \sum_{n \neq i} \frac{\omega_i \ \omega_n}{\omega_i - \omega_n} \ (\bar{X}_i, (G, \delta CU_n)_\Omega)_\Omega (\bar{X}_n, (G, \delta CU_i)_\Omega)_\Omega \tag{22}$$

It is interesting to note that neither the first nor the second variation in the eigenvalue depends upon the particular second normalization factors selected for the eigenfunctions.

APPLICATION

Equations (14) and (22) provide, in very general form, explicit expressions for the first and second variations of the complex frequency for unforced viscously-damped vibrations of elastic systems. Specific usage of the equations will vary with the application. In this section, we shall be concerned with the development of a closed form approximation for ω_i for structures in the presence of "moderate" damping.

Without loss in generality, we may restrict our treatment of damped elastic systems to consideration of the quadratic eigenvalue equation

$$\underline{a}_i = -\omega_i \ \underline{G} \ \underline{C} \ \underline{a}_i - \omega_i^2 \ \underline{G} \ \underline{a}_i \tag{23}$$

where $\underline{G}$ is a diagonal matrix specified in terms of the system's natural frequencies Ω_n. Specifically,

$$G_{ij} = \Omega_i^{-2} \ \delta_{ij} \qquad \text{(no sum)} \tag{24}$$

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Since there are an infinite number of natural frequencies associated with vibrating distributed parameter structures, $\underset{\sim}{G}$ will also contain an infinite number of elements. Likewise, each eigenvector $\underline{a}_i$ is an infinitely dimensional column vector. The matrix $\underset{\sim}{C}$ has elements

$$C_{ij} = (W_i, \ C \ W_j)_\Omega \qquad (25)$$

where W_i is the normalized mode shape corresponding to the (undamped) natural frequency Ω_i.

In order to establish the reduction of Eq. (3) to the form (23), it is necessary to recall the eigenvalue equation

$$W_n = \Omega_n^2 \quad (G, \ M \ W_n)_\Omega \qquad (26)$$

and the orthonormal relations

$$(W_p, \ M \ W_n)_\Omega \ = \ \delta_{pn} \qquad (27)$$

for undamped vibrating elastic structures. It is assumed that Ω_n and W_n are known for the structure under consideration. Since the set of eigenfunctions $\{W_n\}$ form a complete basis, the complex eigenfunction U_i may be expressed

$$U_i = \underset{n}{\Sigma} \alpha_{in} \ W_n \qquad (28)$$

where each α_{ni} is a complex-valued constant. Now substitute Eq. (28) into Eq. (3), take the L_2 inner product of each side of the resulting equation with $M W_p$, change the order of the double integrals that arise and simplify the result using Eqs. (26) and (27). The resulting equation is

$$\alpha_{ip} = - \ \omega_i \ \Omega_p^{-2} \ \underset{n}{\Sigma}\alpha_{in} \quad (W_p, \ C \ W_n)_\Omega - \ \omega_i^2 \ \Omega_p^{-2} \alpha_{ip} \qquad (29)$$

which is precisely Eq. (23) expressed in component form.

It should be pointed out that with similar arguments Eq. (23) can also be established for discrete finite-dimensional eigenvalue problems as well. In this case the L_2-inner product is replaced everywhere by the usual dot product of complex vectors.

We now take up the development of an approximate solution to Eq. (23) using the eigensensitivity theory developed in the previous section of this paper. Toward this end, it is convenient to define a one parameter family of damping matrices $\hat{\underset{\sim}{C}}$ (S) where

$$\hat{\underset{\sim}{C}} \ (S) = \underset{\sim}{C}^D + S \ \Delta\underset{\sim}{C} \qquad (30)$$

in which $\underset{\sim}{C}^D$ contains only the diagonal elements of $\underset{\sim}{C}$ and $\Delta\underset{\sim}{C}$ contains only the off-diagonal elements of $\underset{\sim}{C}$. Thus

$$\underset{\sim}{C}^D \ = \ C_{ij} \ \delta_{ij} \ \text{(no sum)}$$

and

$$\Delta C_{ij} \ = \ \begin{cases} C_{ij} & i \neq j \\ 0 & i = j \end{cases} \qquad (31)$$

Clearly $\hat{\underset{\sim}{C}} \ (1) = C$. Let $\hat{\underline{a}}_i(S)$ and $\hat{\omega}_i(S)$ be, respectively, the eigenvectors and eigenvalues associated with $\hat{\underset{\sim}{C}}(S)$. Now

$$\hat{\omega}_i(S) = \hat{\omega}_i(0) + \hat{\omega}_i{}'(0)S + \tfrac{1}{2}\hat{\omega}_i{}''(0)S^2 + \ \cdots$$

where $\hat{\omega}_i(1)$ is the desired eigenvalue ω_i. Therefore

$$\omega_i = \hat{\omega}_i(0) + \hat{\omega}_i{}'(0) + \tfrac{1}{2}\hat{\omega}_i{}''(0) + \ \cdots\cdots \qquad (32)$$

To compute $\hat{\omega}_i(0)$, set $\underset{\sim}{C} = \underset{\sim}{C}^D$ in Eq. (23), thereby uncoupling the equations. The solutions, obtained immediately, are

$$\hat{\omega}_i(0) = - \ \tfrac{1}{2}C_{ii} \pm \tfrac{1}{2}(C_{ii}{}^2 - 4\Omega_i{}^2)^{\tfrac{1}{2}} \qquad (33)$$

Alternatively, $\hat{\omega}_i(0)$ are the eigenvalues obtained by ignoring the off-diagonal elements in the damping matrix $\underset{\sim}{C}$. Indeed, Hutton (1981) suggests the approximation

$$\omega_i = \hat{\omega}_i(0) \qquad (34)$$

may be used whenever the off-diagonal elements of $\underset{\sim}{C}$ are sufficiently small. Finally, for future reference, it is noted that the eigenvector $\hat{\underline{a}}_i(0)$ associated with $\hat{\omega}_i(0)$ has components which may be taken as the Euclidean unit base vector $\underline{e}_i$, so that

$$\hat{a}_{ip}(0) = \delta_{ip} \ , \qquad \hat{\underline{a}}_i(0) = \underline{e}_i \qquad (35)$$

It now remains to calculate the sensitivity derivatives appearing in Eq. (32). The first derivative $\hat{\omega}'(0)$ is computed directly from Eq. (14) by observing

$$\delta\omega_i = \hat{\omega}_i{}'(0) \ \delta S \ , \qquad \delta C = \Delta\underset{\sim}{C} \ \delta S \qquad (36)$$

Further, a comparison of Eqs. (23) with Eqs. (3), (6) and (11) reveals that M=1 and

$$U_i = \hat{\underline{a}}_i = - \ \hat{\omega}_i \ \underset{\sim}{G} \ \underset{\sim}{C} \ \hat{\underline{a}}_i - \ \hat{\omega}_i^2 \ \underset{\sim}{G} \ \hat{\underline{a}}_i \qquad (37a)$$

$$V_i = \hat{\omega}_i \ \hat{\underline{a}}_i \qquad (37b)$$

$$X_i = - \ \hat{\omega}_i \ \hat{\underset{\sim}{C}} \ \underset{\sim}{G} \ \hat{\underline{X}}_i - \ \hat{\omega}_i^2 \ \underset{\sim}{G} \ \hat{\underline{X}}_i \qquad (37c)$$

$$Y_i = - \ \hat{\omega}_i \ \underset{\sim}{G} \ \hat{\underline{X}}_i \qquad (37d)$$

Since $\hat{\underset{\sim}{C}}(0) = \underset{\sim}{C}^D$ is a diagonal matrix, $\underset{\sim}{G} \ \hat{\underset{\sim}{C}}(0) = \hat{\underset{\sim}{C}}(0)\underset{\sim}{G}$, and Eqns. (35) and (37a,c) imply that

$$X_i(0) = \mu_i \ \underline{e}_i \qquad (38)$$

where μ_i is a complex-valued scalar. It can also be established directly from Eqs. (35) and (37b,d) that

$$Y_i(0) = - \ \mu_i \ \hat{\omega}_i(0) \ \underset{\sim}{G} \ \underline{e}_i$$
$$V_i(0) = - \ \hat{\omega}_i(0) \ \underline{e}_i \qquad (39)$$

The scalar μ_i can be computed from the ortho-normality condition (12) and Eqs. (35), (38), and (39). Thus

$$\mu_i = \Omega_i^2 / [\Omega_i^2 - \hat{\omega}_i^2(0)] \tag{40}$$

After substituting Eqs. (24), (36) and (37a,c) into Eq. (14)

$$\omega_i'(0) = \mu_i \, \hat{\omega}_i^2(0) \, \underline{e}_i \cdot \underline{G} \, \Delta\underline{C} \, \underline{e}_i \tag{41}$$

results. After carrying out the indicated operations in Eq. (41), we obtain

$$\hat{\omega}_i'(0) = \mu_i \, \hat{\omega}_i^2(0)\Omega_i^{-2} \, \Delta \, C_{ii} \equiv 0. \tag{42}$$

Equation (42) validates Hutton's (1981) approximation (34) whenever ΔC is very small, for, in this case, the error in Eq. (34) is of order $(\Delta C)^2$.

It still remains to calculate $\hat{\omega}_i''(0)$ from Eq. (22). We observe that

$$\delta^2\omega_i = \hat{\omega}_i''(0) \, (\delta S)^2$$

and

$$\delta^2 C = 0 \tag{43}$$

Now, we combine Eqs. (22), (24), (33), (35), (36), (37a,c), (40), (41) and (43) to obtain

$$\frac{1}{2} \, \hat{\omega}_i''(0) = \frac{\hat{\omega}_i^3(0)}{\Omega_i^2 - \hat{\omega}_i^2(0)} \sum_{n \neq i} \left[\frac{(\Delta C_{ni})^2}{\hat{\omega}_i(0) - \hat{\omega}_n(0)} \right.$$

$$\left. \cdot \frac{\hat{\omega}_n(0)}{\Omega_n^2 - \hat{\omega}_n^2(0)} \right] \tag{44}$$

Finally, after substituting Eqs. (42) and (44) into Eq. (32), we obtain the result

$$\omega_i = \hat{\omega}_i(0) + \frac{\hat{\omega}_i^3(0)}{\Omega_i^2 - \hat{\omega}_i^2(0)} \sum_{n \neq i} \frac{(\Delta C_{ni})^2}{\hat{\omega}_i(0) - \hat{\omega}_n(0)} \cdot \frac{\hat{\omega}_n(0)}{\Omega_n^2 - \hat{\omega}_n^2(0)}$$

$$\tag{45}$$

where $\hat{\omega}_i(0)$ is given by Eq. (33). If desired, the approximation (45) can be made more accurate by including still higher order derivatives of $\hat{\omega}_i(S)$. In this case, $\delta^3\omega_i$ can be calculated directly from Eqs. (20-22). Although the procedure is straightforward, the final expression for $\delta^3\omega_i$ is cumbersome and consequently will be omitted here.

A few observations regarding the usage of Eq. (45) is in order. The frequencies $\hat{\omega}_i(0)$ always occur in pairs: if they are complex-valued, then the pairs are complex conjugates; but if they are real valued, then the pairs generally consist of distinct negative numbers. When summing over all values of n, other than

n=i, it must be recognized that there are two values for $\hat{\omega}_n(0)$ corresponding to each n which must be included in the summation.

Also, it should be noted that the sensitivity derivatives are evaluated at S=0. These derivatives exist only for distinct zeroth order roots $\hat{\omega}_i(0)$, whether complex or real-valued. And finally, it can be easily shown that the other terms appearing in the denominator of the expression for $\hat{\omega}_i''(0)$, namely $\Omega_j^2 - \hat{\omega}_j^2(0)$ never vanish.

EXAMPLES

(a) As a very simple illustration, consider the elementary non-dimensional quadratic eigenvalue equation corresponding to the two-degree of freedom system in which $C_{11} = 1$, $C_{22} = 2$, $C_{12} = -0.75$, $\Omega_1^2 = 2$, $\Omega_2^2 = 4$. The zeroth order approximation, obtained by ignoring C_{12}, is

$$\hat{\omega}_1(0) = -0.5 \pm 1.9365 \, i$$
$$\hat{\omega}_2(0) = -1.0 \pm 1.0000 \, i \tag{46}$$

On the other hand, the frequencies calculated from the second order approximation (45) are

$$\omega_1 \approx -0.3313 \pm 1.8639 \, i$$
$$\omega_2 \simeq -1.1688 \pm 0.9438 \, i \tag{47}$$

By way of comparison, the exact eigenvalues are

$$\omega_1 = -0.3434 \pm 1.8863 \, i$$
$$\omega_2 = -1.1576 \pm 0.9146 \, i$$

Obviously, the second order approximation (47) is far superior to the zeroth order approximation (46). As a general rule, the most important value is the smallest (in magnitude) real component of ω_i, in this case Re $\{\omega_1\}$. The zeroth order approximation for Re $\{\omega_1\}$ is in error by 50.9%, while the second order approximation for Re $\{\omega_1\}$ is in error by 3.6%.

The accuracy of Eq. (47) is somewhat remarkable considering that C_{12} is not small compared with either of the diagonal elements. However, it can be shown that for systems with two degrees of freedom, Eq. (45) is accurate to within terms of order $(\Delta C)^4$, and hence the rapid convergence of the series approximation (32).

(b) A somewhat more involved quadratic eigenvalue problem is provided by the four degree of freedom system in which $C_{11} = 2$, $C_{22} = 4$, $C_{33} = 1$, $C_{44} = 3$, $C_{ij} = -0.25$ for $i \neq j$, and $\Omega_1^2 = \Omega_2^2 = 2$, $\Omega_3^2 = \Omega_4^2 = 3$. In this example, the off diagonal damping elements are small, but not negligible, compared to the diagonal damping coefficients. There are now four pairs of solutions ω_i, of which one pair

is real-valued and the others are complex-valued. The zeroth order solution, obtained immediately from Eq. (33), is

$$\hat{\omega}_1(0) = -1 \pm i$$
$$\hat{\omega}_2(0) = -0.5858, \quad -3.4142 \qquad (48)$$
$$\hat{\omega}_3(0) = -0.5 \quad \pm \quad 1.6583\,i$$
$$\hat{\omega}_4(0) = -1.5 \quad \pm \quad 0.8660\,i$$

However, using the second order approximation, the solution is

$$\omega_1 \approx -1.0094 \quad \pm \quad 1.0656i$$
$$\omega_2 \approx -0.5718 \quad , \quad -3.5332 \qquad (49)$$
$$\omega_3 \approx -0.4493 \quad \pm \quad 1.6605i$$
$$\omega_4 \approx -1.4888 \quad \pm \quad 0.7616i$$

It can be shown, although with considerably more effort than was required in the two-degree of freedom example, that the exact frequencies are

$$\omega_1 = -1.0142 \quad \pm \quad 1.0506\,i$$
$$\omega_2 = -0.5744 \quad , \quad -3.5062 \qquad (50)$$
$$\omega_3 = -0.4377 \quad \pm \quad 1.6584\,i$$
$$\omega_4 = -1.5077 \quad \pm \quad 0.7591\,i$$

Once again, the largest error in the zeroth order frequency occurs in the least damped frequency - in this case, Re $\{\omega_3\}$. Here, the quadratic approximation produces an error of 2.7% compared with 14.2% for the zeroth order approximation.

(c) As an illustration of a continuous system, consider the unforced vibrations of a uniform cantilever beam subject to distributed damping. Specifically, the quadratic eigenvalue equation is

$$EIU^{IV}(x) + C(x)\omega U(x) + \rho\omega^2 U(x) = 0 \qquad (51a)$$

with boundary conditions

$$U(0) = U'(0) = U''(L) = U'''(L) = 0 \qquad (51b)$$

It will be assumed that uniform damping occurs over the outer half of the beam, so that

$$C(x) = C_o H(x-L/2) \qquad (52)$$

where C_o is the damping coefficient and H is the Heaviside function. Further, EI is the beam's uniform flexural stiffness, L is its length, ρ is its mass density per unit length and U is the response eigenfunction. It is convenient to cast Eq. (51a) into non-dimensional form by setting

$$Z = U/L \quad , \quad y = x/L, \quad a^2 = L^4\,[\rho\omega^2/EI]$$

$$\beta = C/2\sqrt{\rho EI} \quad , \quad \beta_o = C_o/2\sqrt{\rho EI} \qquad (53)$$

so that Eqs. (51a,b) and Eq. (52) become

$$z^{IV} + 2\beta a z + a^2 z = 0$$
$$Z(0) = Z'(0) = Z''(1) = Z'''(1) = 0 \qquad (54)$$
$$\beta = \beta_o H(y - 1/2)$$

Here, a is the desired eigenvalue, Z the corresponding eigenfunction, and 2β is the damping operator.

In order to use the quadratic approximation (45), Eq. (54) must be cast into the form (23). It is therefore necessary to first obtain the natural frequencies Ω_n and corresponding mode shapes $W_n(y)$ for the undamped system, i.e. $\beta = 0$. These results, however, are readily available. If we let

$$\eta_{ij} = \int_{\frac{1}{2}}^{1} W_i(y)\,W_j(y)\,dy \qquad (55)$$

the damping matrix becomes

$$C_{ij} = 2\,\beta_o\,\eta_{ij} \qquad (56)$$

Direct computations of η_{ij} provide specific values of C_{ij}. The zeroth order frequencies now become

$$\hat{a}_i(0) = -\beta_o\eta_{ii} \pm i\sqrt{\Omega_i^2 - \beta_o^2\eta_{ii}^2} \qquad (57)$$

while the second order approximation to the frequencies can be obtained from Eq. (44) using specific values for β_o.

Results for the first six eigenvalues are presented in Table 1 for light damping $\beta_o = 1.5$ and moderate damping $\beta_o = 3.0$. For $\beta_o = 1.5$, the zeroth order frequencies (57) are very close to the more accurate second order approximation, denoted by a_i^*. The maximum difference in the real parts of the frequencies occurs in a_3, the least damped mode, and is approximately 4%.

For $\beta_o = 3.$, the difference between $\hat{a}_i(0)$ and a_i^* is more pronounced. As shown in Table 2, both eigenvalues corresponding to a_1 are real, while the remaining ones are complex-valued. Except for the second and third frequencies, the quadratic approximation a_i^* adds relatively little to the much simpler solution $\hat{a}_i(0)$. But the Re $\{\hat{a}_3(0)\}$ predicts 16% more damping than Re $\{a_3^*\}$, while the Re $\{\hat{a}_2(0)\}$ predicts 10% less damping than Re $\{a_2^*\}$.

In the absence of an exact solution to Eq. (54), it is difficult to precisely determine the error introduced by truncating the series (32) after quadratic terms. The exact solution is the limiting case, as $N \to \infty$, of an N degree of freedom system obtained by considering the sub-matrices of $\underset{\sim}{C}$ and $\underset{\sim}{G}$ in which i, j=1,...., N. Unfortunately, it is difficult to solve, exactly, the quadratic eigenvalue problem when N is not a small number. Nevertheless, by examining the exact eigenvalues for the finite dimensional problem for small values of N, an assessment of the relative accuracy of a_i^* can be made.

TABLE 1: ZEROTH AND SECOND ORDER EIGENVALUES FOR $\beta \cdot = 1.5$

i	$\hat{a}_i(0)$	a_i^*
1	$-1.424 \pm 1.220i$	$-1.441 \pm 1.216i$
2	$-0.838 \pm 4.619i$	$-0.858 \pm 4.684i$
3	$-0.739 \pm 7.820i$	$-0.712 \pm 7.716i$
4	$-0.748 \pm 10.970i$	$-0.752 \pm 10.996i$
5	$-0.750 \pm 14.117i$	$-0.744 \pm 14.058i$
6	$-0.750 \pm 17.262i$	$-0.752 \pm 17.279i$
7	$-0.750 \pm 20.407i$	$-0.747 \pm 20.365i$

TABLE 2: ZEROTH AND SECOND ORDER EIGENVALUES FOR $\beta \cdot = 3.$

i	$\hat{a}_i(0)$	a_i^*
1	$-4.991, \quad -0.704$	$-5.187, \quad -0.699$
2	$-1.676 \pm 4.385i$	$-1.867 \pm 4.623i$
3	$-1.479 \pm 7.714i$	$-1.273 \pm 7.341i$
4	$-1.497 \pm 10.893i$	$-1.526 \pm 10.994i$
5	$-1.501 \pm 14.057i$	$-1.449 \pm 13.824i$
6	$-1.500 \pm 17.213i$	$-1.511 \pm 17.277i$
7	$-1.500 \pm 20.365i$	$-1.475 \pm 20.202i$

TABLE 3: EXACT EIGENVALUES FOR N-DEGREES OF FREEDOM, $\beta \cdot = 3.0$

	N=2	N=3	N=4	N=5	N=6
a_1	$-5.137,$ -0.700	-5.147 -0.699	-5.150 -0.699	-5.151 -0.699	-5.152 -0.699
$Re(a_2)$	-1.605	-1.934	-1.925	-1.942	-1.942
$Im(a_2)$	4.355	4.659	4.661	4.673	4.673
$Re(a_3)$	$-$	-1.145	-1.208	-1.188	-1.190
$Im(a_3)$	$-$	7.132	7.273	7.258	7.261
$Re(a_4)$	$-$	$-$	-1.442	-1.569	-1.559
$Im(a_4)$	$-$	$-$	10.678	11.047	11.033
$Re(a_5)$	$-$	$-$	$-$	-1.376	-1.429
$Im(a_5)$	$-$	$-$	$-$	13.566	13.782
$Re(a_6)$	$-$	$-$	$-$	$-$	-1.455
$Im(a_6)$	$-$	$-$	$-$	$-$	16.957

The exact frequencies for the reduced N-degree of freedom problem, obtained by solving the characteristic polynomial equation of order 2N, are listed in Table 3 for N=2 through N=6. A cursory examination of the values for a_1, a_4, a_5 and a_6 suggests that the corresponding values of the second order approximation a_i^*, shown in Table 2, are very reasonable estimates of the exact distributed parameter eigenvalues. Table 3 also suggests that the exact eigenvalues for a_2 and a_3 are $-1.94 \pm 4.67i$ and $-1.19 \pm 7.26i$, respectively. Thus the real part of these zeroth order approximations differ by 14% and 24% from the exact values, while the second order approximations produce errors of only 4% and 7%, respectively. Similarly, the error in the imaginary parts of a_2 and a_3 drops from 6% for the zeroth order approximation to about 1% for the second order approximation.

SUMMARY

We have presented an approximate solution to the quadratic eigenvalue problem by expanding the eigenvalues in a series about $\Delta \underset{\sim}{C} = 0$ and truncating after quadratic terms. It was shown, by examples, that whenever $\Delta \underset{\sim}{C}$ is small but not negligible, the series solution (45) provides reasonable estimates of the true eigenvalues even when the solution obtained by ignoring $\Delta \underset{\sim}{C}$ is substantially in error.

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5. ON THE FUNDAMENTAL FREQUENCY OF SYMMETRIC RECTANGULAR LAMINATED PLATES: A NEW CLOSED-FORM APPROXIMATE SOLUTION

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ABSTRACT

A new closed-form approximation for the fundamental frequency of symmetric rectangular laminates subject to all combinations of hinged and clamped boundary conditions is presented. The distributed parameter eigenvalue equation is cast in an equivalent infinitely-dimensional discrete form. The stiffness and mass matrices are each decomposed into the sum of two matrices, one of which is diagonal while the other contains zero diagonal elements. Design sensitivity analysis is used to expand the desired eigenfrequency in a Maclaurin series of the zero diagonal matrices. The general formula thus obtained is then specialized to rectangular symmetric laminated plates. The remarkable occuracy of this new formula is established by numerical comparisons of results calculated from it to those obtained from the conventional Rayleigh-Ritz method.

INTRODUCTION

Methods to accurately determine the frequencies of laminated plates is necessary for both analysis and design of symmetric laminated composite plates. While exact closed form solutions for the natural frequencies are available only for simply-supported orthotropic laminates [1], very accurate approximate solutions are also available for orthotropic laminates meeting other boundary conditions [2].

For more general anisotropic plates, the literature contains several closed form approximations for the fundamental frequency. Bert obtained simple formula for symmetric rectangular laminates for simply supported [3] and clamped edges [4] by assuming a linear relationship between the eigenvalue and the flexural stiffnesses. A different approach was adopted by Reiss and coworkers. Reiss, Ramachandran and Qian [5], using exact sensitivity derivatives, obtained an approximate solution for the natural frequencies of a four-layer symmetric simply-supported angle-ply laminate. The frequencies obtained were presented in the form of a truncated Fourier series in the ply-angle. Reiss and Qian [6,7] then generalized those results to include all symmetric laminates, none of whose edges were free.

In the present study, a new and much more general eigensensitivity-based method for obtaining approximate formula for the frequency response of structures is employed. The method, recently developed by Reiss, Qian and Aung [8] for damped systems and Reiss and Qian [9] for self-adjoint systems requires recasting the distributed parameter equation into an equivalent discrete set of algebraic equations. The new formula for the fundamental frequency of symmetric laminates is obtained by specializing the general result [9] to the specific problem under consideration.

REVIEW OF PREVIOUS RESULTS

At this point, it is useful to review the general methodology previously developed by Reiss and Qian [9].

The general eigenvalue problem under consideration is representable in the form

$$T^*ETu_i = \lambda_i Mu_i \quad in \quad \Omega \qquad (1)$$

subject to appropriate mixed boundary conditions. Here T and T^* are $L_2(\Omega)$ adjoint differential operators, E and M are, respectively, the stiffness and mass operators which are each positive and symmetric with respect to the $L_2(\Omega)$ inner product. Also λ_i is the eigenvalue associated with the eigenfunction u_i.

While there are many ways in which Equation (1) may be converted into a discrete form, the Raleigh-Ritz method is particular useful in view of the objectives of this paper. Thus, let $\{\psi_n\}$ denote a complete set of kinematically admissible basis functions, so that

$$u_i = \sum_{n=1}^{\infty} \alpha_{in} \psi_n \qquad (2)$$

Stationarily of the Rayleigh quotient requires

$$[K]\{\alpha_i\} = \lambda_i [M]\{\alpha_i\} \qquad (3)$$

where the square, infinitely dimensional matrices $[K]$ and $[M]$ contain elements

$$K_{ij} = (T\psi_i, ET\psi_j)$$

$$ \qquad (4)$$

$$M_{ij} = (\psi_i, M\psi_j)$$

In Equation (4), $(\cdot, \cdot)$ denotes the $L_2(\Omega)$ inner product. Further α_{in} is the n-th component of the infinitely dimensional column vector $\{\alpha_i\}$.

By introducing new matrices $[K^0]$, $[k]$, $[M^0]$ and $[m]$ defined by

$$K_{ij}^0 = K_{ij} \delta_{ij} \qquad \text{(no sum)}$$

$$M_{ij}^0 = M_{ij} \delta_{ij} \qquad \text{(no sum)} \qquad (5)$$

$$k_{ij} = K_{ij} - K_{ij} \delta_{ij} \qquad \text{(no sum)}$$

$$m_{ij} = M_{ij} - M_{ij} \delta_{ij} \qquad \text{(no sum)}$$

where δ_{ij} is the Kronecker delta, it follows that Equation (3) becomes

$$([K^0] + [k])\{\alpha_i\} = \lambda_i([M^0] + [m])\{\alpha_i\}$$

$$ \qquad (6)$$

In view of Equation (5), it is evident the $[K^0]$ and $[M^0]$ are diagonal matrices, while $[k]$ and $[m]$ contain only zero values on the diagonal. It is convenient to consider λ_i to be a scalar-valued function of the arguments $[k]$ and $[m]$. Indeed, if λ_i is expanded as a Maclaurin series in these arguments, it can be shown that λ_i satisfies [9]

$$\lambda_i = K_{ii} M_{ii}^{-1} - M_{ii}^{-2} \sum_{n \neq i}^{\infty} \frac{(M_{ii} k_{ni} - K_{ii} m_{ni})^2}{M_{ii} K_{nn} - M_{nn} K_{ii}}$$

$$ \qquad (7)$$

Equation (7) is exact to within the order of terms retained in the series. Higher order terms in the expansion may be readily calculated; however, only by compromising the simplicity of Equation (7).

SYMMETRIC LAMINATED PLATES

The eigenvalue equation for a freely vibrating symmetric laminate is

$$D_{11}u_{xxxx} + D_{22}u_{yyyy} + 2(D_{12} + 2D_{66}) u_{xxyy}$$

$$ + 4 D_{16}u_{xxxy} + 4 D_{26} u_{xyyy} = \rho \omega^2 u$$

$$ \qquad (8)$$

where u denotes the mode shape, ω is the corresponding frequency, ρ is the mass density, D_{ij} are the flexural stiffnesses [1], x and y are the coordinates in the plane of the laminate, and indices following u denote differentiation with respect to the indicated argument.

In order to apply Equation (7), it is necessary to cast Equation (8) in the form of Equation (1), that is, the operators T, T^*, E, M and the basis functions $\{\psi_n\}$ must be identified. It is straight-forward to show that

$$T = \begin{bmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ 2\partial^2/\partial x \partial y \end{bmatrix}, \quad T^* = T', \quad M = \rho,$$

$$\lambda = \omega^2 \qquad (9)$$

and, since no edges are free, the stiffness operator may be selected as

$$E = \begin{bmatrix} D_{11} & D_e & D_{16} \\ D_e & D_{22} & D_{26} \\ D_{16} & D_{26} & 0 \end{bmatrix} \qquad (10)$$

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where

$$D_e = D_{12} + 2 D_{66}$$

As is common in plate theory, the basis functions will be selected as products of the beam shape functions, i.e.

$$\psi_{mn}(x,y) = \rho X_m(x/a) Y_n(y/b) \quad (11)$$

where the plate domain is $0 \le x \le a$, $0 \le y \le b$ and $R = a/b$ is the laminate's aspect ratio. The beam shape functions satisfy

$$X_m^{IV} - \mu_m^4 X_m = 0 \quad 0 \le x/a \le 1$$

$$Y_n^{IV} - v_n^4 Y_n = 0 \quad 0 \le y/b \le 1$$

where $(\ )^{IV}$ denotes the fourth derivative with respect to the indicated non-dimensional argument. It is worth noting that X_m, Y_n, μ_m and v_n are known for each set of boundary conditions. Further, if the basis functions are normalized with respect to the mass density ρ, it follows that the shape functions must satisfy

$$(X_i, X_j) = (Y_i, Y_j) = \rho^{-1/2} \delta_{ij} \quad (12)$$

It remains to determine the mass and stiffness elements from Equation (4). Since the basis functions in Eq. (11) are double subscripted, it follows that the proper form for Eq. (4) is

$$K_{ijmn} = (T \psi_{ij}, BT \psi_{mn})$$
$$M_{ijmn} = (\psi_{ij}, M \psi_{mn}) \quad (13)$$

while Equation (7) becomes

$$\lambda_{11} = \frac{K_{11111}}{M_{1111}} - \frac{1}{M_{1111}^2} \Sigma\Sigma \frac{(M_{1111}k_{mn11} - K_{1111}m_{mn11})^2}{M_{1111}K_{mnmn} - M_{mnmn}K_{1111}} \quad (14)$$

Substitution of Equations (9-11) into (13) readily yields

$$M_{ijmn} = \delta_{im} \delta_{jn} \quad (15)$$

$$\rho a^4 K_{ijmn} = (D_{11}\mu_i^4 + R^4 D_{22} v_j^4) \delta_{im} \delta_{jn}$$
$$+ 2D_e R^2 C_{im}C_{jn} + 2RD_{16}g_{nj} (H_{im} - H_{mi})$$
$$+ 2R^3 D_{26} G_{mi} (h_{jn} - h_{nj}) \quad (16)$$

where

$$C_{ij} = (X_i', X_j') \qquad c_{ij} = (Y_i', Y_j')$$
$$G_{ij} = (X_i', X_j) \qquad g_{ij} = (Y_i', Y_j)$$
$$H_{ij} = (X_i'', X_j') \qquad h_{ij} = (Y_i'', Y_j')$$

In view of Equations (15-16), the terms appearing in Equation (14) are

$$\rho a^4 K_{1111} = D_{11}\mu_1^4 + R^4 D_{22}v_1^4 + 2D_e R^2 C_{11}C_{11}$$

$$M_{1111} = M_{mnmn} = 1$$

$$\rho a^4 K_{mnmn} = D_{11}\mu_m^4 + R^4 D_{22}v_n^4 + 2D_e R^2 C_{mm}c_{nn}$$

$$\rho a^4 k_{mn11} = 2D_e R^2 C_{m1}c_{n1} + 2RD_{16}g_{1n} (H_{m1} - H_{1m})$$
$$+ 2R^3 D_{26}G_{m1} (h_{1n} - h_{n1})$$

$$m_{mn11} = 0 \quad (17)$$

Finally, substitution of Equations (17) into (14) produces the desired result

$$\rho a^4 \lambda_{11} = \rho a^4 \omega_{11}^2 = D_{11}\mu_1^4 + R^4 D_{22}v_1^4 + 2D_e R^2 C_{11}C_{11}$$
$$- 4R^2 \sum_{(m,n \ne 1)} \sum \{D_e RC_{m1}c_{n1} + D_{16}g_{1n} (H_{m1} - H_{1m})$$
$$+ D_{26}R^2 G_{m1} (h_{1n} - h_{n1})\}^2 / \{D_{11}(\mu_m^4 - \mu_1^4)$$
$$+ D_{22}R^4 (v_n^4 - v_1^4) + 2D_e R^2 (C_{mm}c_{nn} - C_{11}C_{11})\}$$

$$(18)$$

DISCUSSION

Equation (18) provides the desired approximation for the fundamental frequency of laminated rectangular plates subject to all combinations and clamped and simply-supported boundary conditions. In the derivations of this equation, it was implicitly assumed that the natural frequency ω_{11} is in fact the fundamental frequency. It can be established that for the boundary conditions under consideration, this assumption is indeed correct. Finally, it should be pointed out that the matrices [C], [c], [G], [g], [H] and [h] and beam frequencies μ_i and v_i are boundary condition dependent; hence these values account for the effect of the boundary conditions in Equation (18).

Apart from the term proportional to D_e in the numerator of the double sum term appearing in Equation (18), the same expression for ω_{11} was obtained by Reiss and Qian [7]. In their approach, ω_{11} was treated as a function of D_{16} and D_{26}, and the sensitivity derivatives of ω_{11} with respect to these arguments were used to generate the approximate solution.

However, since these derivatives could not be exactly evaluated, a somewhat hueristic approach was used to obtain the ultimate result. In the approach of this paper, however, exact sensitivity derivatives have been obtained, and consequently Equation (18) is exact to within the order of terms retained in the Maclaurin series.

RESULTS

It is noted that a double infinite sum appears in Equation (18). Of course, in practice, only finite sums may be evaluated. If, say, m and n take on values up to N, then, strictly speaking, Equation (18) provides an approximate solution to the finite-dimensional Rayleigh-Ritz equation in which the matrices [M] and [K] consist of N^2 rows and columns. Consequently in assessing the accuracy of Equation (18), it is necessary to compare the approximate solution for a given N to the Rayleigh-Ritz solution for the same N.

In the following tables, the validity of Equation (18) is established by comparing the fundamental frequency calculated from it to the corresponding frequency obtained from the Rayleigh-Ritz approach. In all cases N was selected to be nine; thus there are 81 terms in the expression for the lowest mode shape u_{11}. Results are presented only for balanced symmetric four-ply plates [θ/-θ/-θ/θ], since the addition of more layers reduces the effect of the coupling terms D_{16} and D_{26}. For comparison purposes, it is convenient to introduce the fundamental frequency

$$k_1 = \omega_{11}b^2\sqrt{\rho/U_1 h^3}$$

where U_1 is an invariant material property [1] and h is the common thickness of the plies.

Each table includes results for typical boron epoxy (B-E) and high modulus graphite epoxy (G-E) laminates defined by the following data:

	B-E	G-E
E_1/E_2	10	40
E_1/G_{12}	40	80
ν_{12}	0.30	0.25

The tables have been developed for various aspect ratios and boundary conditions. The nomenclature S and C is used for simply-supported and clamped edges, respectively. The first letter denotes the support condition on the edge x=0; the next for y=0; the third for x=a; and

finally the edge y=b. The columns labelled k_1 are calculated using the Ritz method with N=9, while the column labelled k_1 is determined from Equation (18) also with N=9. Finally, since the frequency is symmetric in θ, results are limited to ply angles between 0° and 90°.

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TABLE I   RESULTS FOR THE SSSS PLATE

ϕ^o	R	$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
		B-E		C-E	
0	1	38.895	38.895	37.954	37.954
15	1	41.010	41.075	39.815	39.991
30	1	45.509	45.782	44.338	44.957
45	1	47.789	48.136	46.784	47.524
60	1	45.509	45.782	44.338	44.957
75	1	41.010	41.075	39.815	39.991
90	1	38.895	38.895	37.954	37.954
0	2	15.590	15.590	11.687	11.687
15	2	18.117	18.129	14.945	15.023
36	2	23.139	23.215	21.236	21.492
45	2	27.784	27.945	27.023	27.384
60	2	31.560	31.688	31.803	32.042
75	2	34.362	34.383	35.341	35.382
90	2	35.451	35.451	36.707	36.707
0	5	11.389	11.389	6.208	6.208
15	5	12.151	12.151	7.859	7.866
30	5	15.178	15.187	12.719	12.778
45	5	20.831	20.859	20.242	20.307
60	5	27.540	27.557	28.244	28.272
75	5	32.803	32.806	34.234	34.339
90	5	34.776	34.776	36.442	36.442

TABLE II   RESULTS FOR THE CCCC PLATE

ϕ^o	R	B-E $\bar{k}_1$	k_1	C-E $\bar{k}_1$	k_1
0	1	84.172	84.172	84.284	84.284
15	1	82.260	82.181	81.592	81.467
30	1	78.658	78.355	76.642	76.416
45	1	77.014	76.580	74.474	74.216
60	1	78.658	78.355	76.642	76.416
75	1	82.260	82.181	81.592	81.467
90	1	84.172	84.172	84.284	84.284
0	2	32.817	32.816	25.050	25.050
15	2	34.273	34.241	26.997	26.942
30	2	39.773	39.621	34.846	34.704
45	2	50.513	50.328	48.936	48.782
60	2	64.002	63.861	65.210	65.050
75	2	75.055	75.010	77.963	77.898
90	2	79.275	79.275	82.756	82.756
0	5	25.276	25.276	13.644	13.644
15	5	25.892	25.894	15.603	15.590
30	5	31.498	31.478	25.746	25.688
45	5	44.918	44.872	43.749	43.682
60	5	61.212	61.178	62.955	62.912
75	5	73.911	73.901	77.268	77.256
90	5	78.651	78.652	82.536	82.536

TABLE III   RESULTS FOR THE CSCS PLATE

θ°	R	B-E		G-E	
		$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
0	1	80.824	80.824	83.327	83.327
15	1	78.613	78.637	80.628	80.686
30	1	74.283	74.276	75.494	75.500
45	1	67.609	67.638	67.565	67.621
60	1	57.428	57.481	55.133	55.236
75	1	48.131	48.134	43.346	43.318
90	1	45.333	45.333	39.957	39.957
0	2	23.778	23.778	22.007	22.007
15	2	25.280	25.275	23.835	23.833
30	2	29.055	29.077	28.423	28.461
45	2	31.804	31.834	31.606	31.653
60	2	33.402	33.348	33.547	33.477
75	2	35.097	35.051	35.704	35.625
90	2	36.027	36.027	36.879	36.880
0	5	11.806	11.806	6.935	6.935
15	5	12.636	12.641	8.709	8.716
30	5	15.679	15.682	13.445	13.435
45	5	21.121	21.077	20.573	20.505
60	5	27.647	27.599	28.344	28.279
75	5	32.842	32.826	34.255	34.234
90	5	34.803	34.804	36.451	36.452

TABLE IV   RESULTS FOR THE SCCC PLATE

θ°	R	B-E		G-E	
		$\bar{k}_1$	k_1	$\bar{k}_1$	k_1
0	1	61.839	61.839	59.340	59.340
15	1	62.322	62.255	59.334	59.221
30	1	64.755	64.535	61.623	61.515
45	1	69.466	69.254	67.296	67.322
60	1	75.034	74.965	74.264	74.380
75	1	80.008	79.985	80.820	80.813
90	1	82.104	82.105	83.701	83.702
0	2	29.487	29.487	20.052	20.052
15	2	31.377	31.353	22.997	22.964
30	2	37.934	37.864	32.769	32.757
45	2	49.665	49.645	48.174	48.242
60	2	63.658	63.652	64.991	65.011
75	2	74.871	74.859	77.895	77.883
90	2	79.116	79.118	82.710	82.711
0	5	25.158	25.158	13.420	13.420
15	5	25.760	25.764	15.388	15.384
30	5	31.399	31.404	25.643	25.644
45	5	44.872	44.872	43.711	43.709
60	5	61.192	61.189	62.941	62.936
75	5	73.902	73.900	77.264	77.262
90	5	78.644	78.646	82.534	82.534

6.   THE LOAD CORRECTION PRINCIPLE

Taft H. Broome, Jr. Sc.D.

ABSTRACT

A methodology for generating a geometrically general class of continua to represent large lattices is presented. This methodology is characterized by a mapping of the geometry and material properties of the lattice into general continua whose material properties can be prescribed independently of the lattice's architecture, and whose loading consists of the lattice's loading together with an intrinsic load called the "load correction." This load correction is formulated in a manner that is consistent with the constraint that the continuum and the lattice exhibit similar kinematic responses to their respective loadings. Knowledge about the lattice is obtained from a Ritz analysis of the continuum wherein efficiency derives from the repeating nature of the lattice's architecture and the small set of basis functions necessary to describe the global response of the continuum.

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1. Introduction

The purpose of this study is to assess the utility of the load correction principle[1] by which the mechanics of large discrete structural systems can be mapped into the mechanics of continuous media. The aim is to contribute new experiences toward the production of a comprehensive theory that can be implemented economically for the analysis of large complex lattice systems.

Since lattices will likely provide structural support for space stations, large orbiting antennas, solar power satellites, etc., then miles of cosmic space may be occupied by individual periodic assemblages of structural elements possessing identical architectures (see Figure 1). Lattices of this sort possess enormous numbers of degrees of freedom ($>10^4$ d.o.f). Such numbers are sufficient to eliminate the finite element method (FEM) from contention as an economical tool for the global analysis of large space structures (LSS). Thus, more economical analysis methods, such as "continuum modeling" methods, offer the promise of economical alternatives to global application of the FEM to LSS.

One class of continuum modeling methods focuses upon similarities - where they exist - between the material domains of lattices and simple continua such as rods, beams, shafts, plates, and shells[2]. Material properties for a continuum are sought that enable it and the lattice it attempts to model, to exhibit similar kinematic responses to similar loadings while being constrained by similar displacement boundary conditions. Thus, the large set of algebraic equations associated with FEM models of LSS are avoided in favor of small sets of differential equations to which closed-form solutions exist,

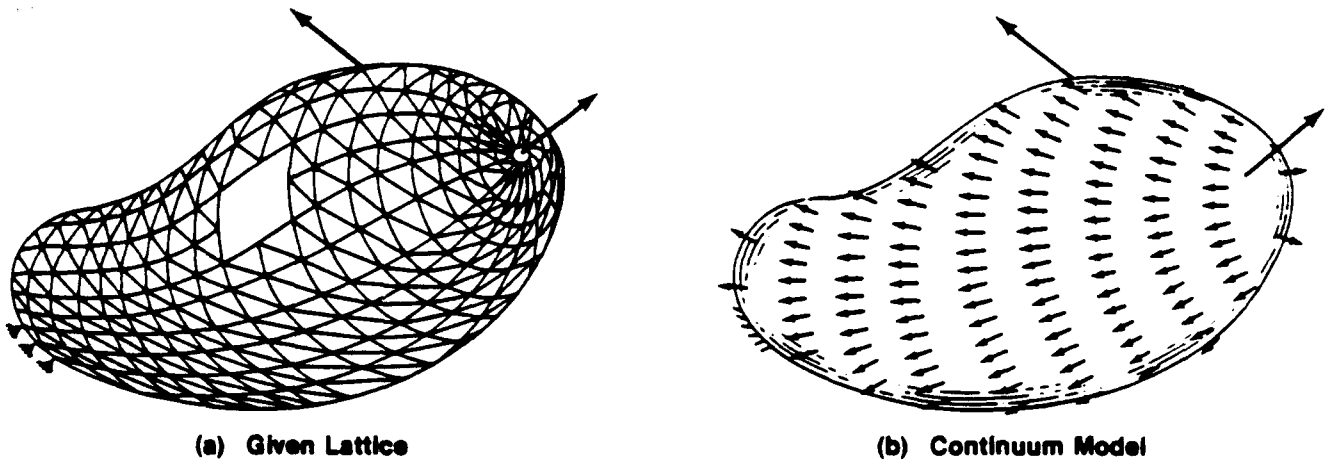(a) Given Lattice

(b) Continuum Model

Figure 1: Given Lattice and Continuum Model

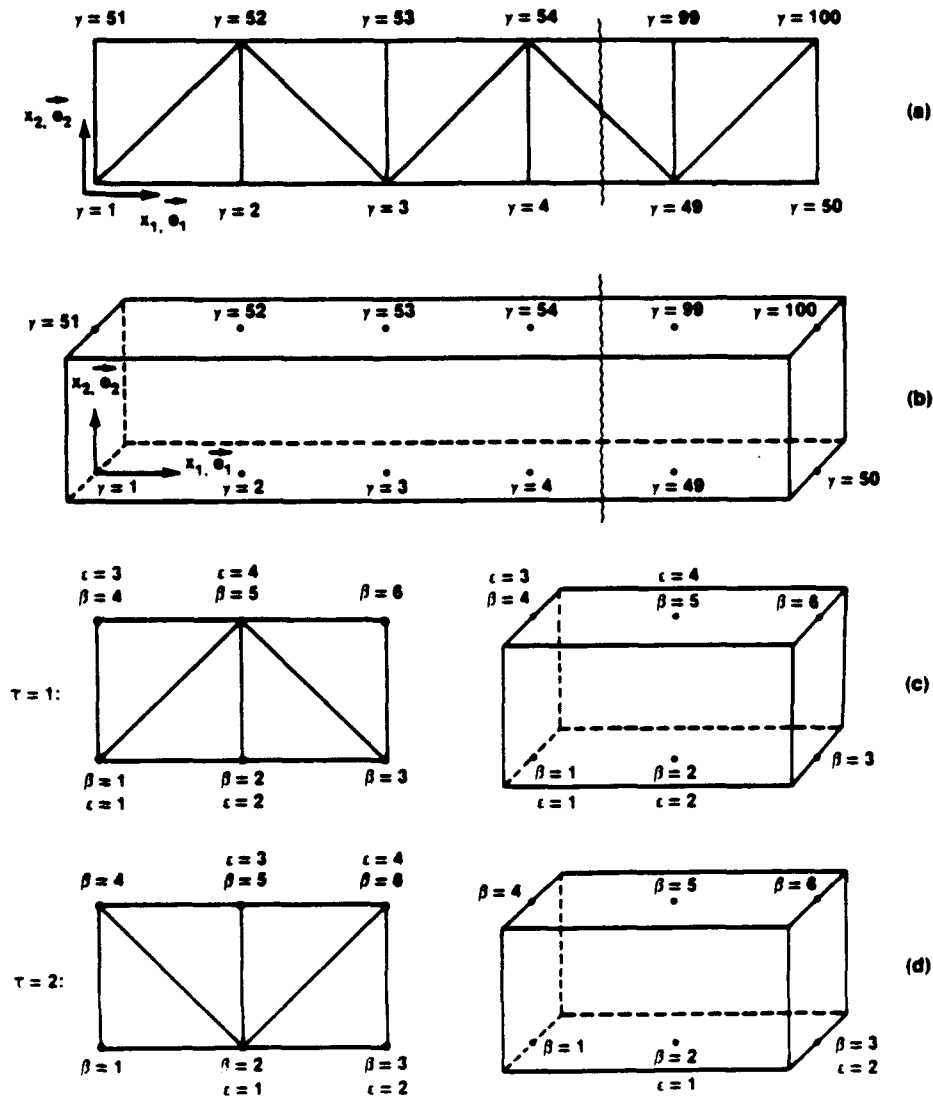**Figure 2: Numbering Schemes for Undeformed Domains.**
(a) The Given Lattice;
(b) The Continuum Model;
(c) Lattice and Continuum Clusters of Type $\tau = 1$; and
(d) Lattice and Continuum Clusters of Type $\tau = 2$.

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or conventional solution techniques are applicable. The load correction method (LCM), which is a means of implementing the load correction principle, is a straight forward general continuum modeling method which is adaptable to both simple and complex domains.

The LCM permits _a priori_ specifications of the material properties of the continuum model, and seeks a new loading for it that is consistent with the constraint that the model and the lattice exhibit similar kinematic responses (including observance of similar kinematic displacement boundary conditions).[3] As we shall see, this new loading consists of the loading applied to the lattice and a displacement-dependent load called the "load correction." Because closed-form solutions are not yet readily apparent, a Ritz analysis is used to transform the large set of algebraic equations produced by the FEM into a smaller, more manageable set. Also, the inner-product nature of a Ritz analysis enables a Dirac-delta technique to take advantage of the repeating architecture of the LSS to arrive at this small set of algebraic equations economically.

The objectives of this study are as follows:

(a) Introduce a nomenclature that facilitates the uncoupling of globally dependent parameters from locally, or what will be defined as "cluster" dependent parameters;

(b) Formulate the given lattice problem;

(c) Map the given lattice problem into a continuum model according to the load correction principle;

(d) Solve the mode';

(e) Illustrate how the globally dependent parameters can be evaluated

economically; and

(f) Demonstrate the utility of the LCM via sample problems.

The utility of the LCM will be assessed in terms of comparisons with the FEM and an LCM/FEM hybrid approach; and in terms of the degree to which the applications spectrum of the LCM extends beyond those of other continuum modeling methods.

The scope of this study is limited to the static analysis of planar lattices. This is the narrowest scope within which the objectives of this study can be realized, and it is adequate as an introduction to the LCM. Moreover, even though modal analysis of LSS is of immediate interest to dynamic control efforts, the role of the LCM in these efforts will likely take the form of a static tool for generating generalized or reduced-order flexibility matrices for LSS. Thus, this study is the first stage in the development of a comprehensive LCM theory.

The remainder of this study is organized into seven sections and three appendices. The following six sections address the six objectives respectively, and the seventh summarizes them with conclusions about the utility of the LCM. The appendices contain a list of the LCM's nomenclature, definitions of some analysis parameters, and a list of references.

2. The Cluster Concept

The given lattice structure will be modeled by a continuum whose material domain spans a configuration of points that is always identical to the configuration of the lattice's joints (for example, see Figures 2a and b). These points in the continuum are called the "images" of the joints in the lattice.

The index "γ" identifies each joint in the lattice as well as its corresponding image in the continuum. Augmenting this global numbering system is a nomenclature that associates each joint and its image with locally repeating architectures called "clusters."

A <u>lattice</u> cluster is defined as a subdomain of the given lattice that facilitates the construction of a stiffness equation relating the external forces at a joint to the translational displacements of this joint and of other neighboring joints. This locally dependent equation will be called the "cluster stiffness" equation. From the examples of lattice clusters shown in Figures 2c and d, it is clear that the clusters are themselves assemblages of repeating elements (or cells) that have been the focus of analysis in other continuum modeling approaches.

Similarly, the images of the joints defining a lattice cluster will define its associated <u>continuum</u> cluster.

A set of identical lattice clusters, each being associated with identical continuum clusters, are identified collectively by the index "τ". In Figure 2c, for example, the clusters bounded by global sets $\{\gamma = 1,2,3,51,52,53\}$, $\{\gamma = 3,4,5,53,54,55\}$, etc, are identified as clusters of type $\tau = 1$. Sets $\{\gamma = 2,3,4,52,53,54\}$, $\{\gamma = 4,5,6,54,55,56\}$, etc., are identified as clusters of type $\tau = 2$.

With respect to each cluster type, the joints and images at which the forces in the cluster stiffness equations are applied are called "principal" joints and images, respectively, in the cluster, and are identified by the index "ε." Moreover, the joints and images at which the displacements in these equations are required are identified by the index "β." The index "α"

is used to distinguish two or more principal joints or images sharing the same (τ, ε) designation, but different γ designations. We define α so that there is a one-to-one relationship between each γ and each 3-tuple $(\tau, \varepsilon, \alpha)$. Symbolically,

$$\gamma = \mathbf{g}(\tau, \varepsilon, \alpha) \; , \; (\tau, \varepsilon, \alpha) = \mathbf{g}^{-1}(\gamma) \qquad\qquad 2.1$$

where: $\mathbf{g}$ is an operator; and $\mathbf{g}^{-1}$ is its unique inverse (see Table 1).

The 4-tuple $(\tau, \varepsilon, \alpha, \beta)$ identifies a unique point γ in a global domain according to the rule

$$\gamma = \mathbf{G}(\tau, \varepsilon, \alpha, \beta) \; , \; (\tau, \varepsilon, \alpha, \beta) = \mathbf{G}^{-1}(\gamma) \qquad\qquad 2.2$$

where: $\mathbf{G}$ is an operator; and its inverse $\mathbf{G}^{-1}$ exists, but is not necessarily unique. We define $\mathbf{G}$ so that the spacial relationship shared by points $\mathbf{g}(\tau, \varepsilon, \alpha)$ and $\mathbf{G}(\tau, \varepsilon, \alpha, \beta)$ in the global domain is identical to the spatial relationship shared by points identified by the pairs (τ, ε) and (τ, β) in the clusters, respectively (see Table 2).

Figures 2c and d, together with Tables 1 and 2, define a cluster "scenario" by which locally dependent stiffness equations can be constructed at each joint in the lattice, and stiffness equations of similar forms can be constructed at the images of these joints in the continuum. Other scenarios are possible - some affording increased precision in the cluster stiffness equations, but at the expense of increased computational requirements.

In the analysis which follows, the Greek indices τ, ε, α and β are used as superscripts to distinguish discrete parameters from their continuous forms (see Table 3). Roman subscripted indexes m, n, i and j denote components of

TABLE 1: The g-transformation for Figure 2.

τ	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
ε	1	2	2	2	-	2	3	4	4	4	-	4	1	1	1	-	1	2	3	3	3	-	3	4
α	1	1	2	3	-	24	1	1	2	3	-	24	1	2	3	-	24	1	1	2	3	-	24	4
γ	1	2	4	6	-	48	51	52	54	56	-	98	3	5	7	-	49	50	53	55	57	-	99	100

TABLE 2: A sample of the G-transformation for Figure 2.

τ	1	1	1	1	1	1	1	1	1	1	1	1	1
ε	1	2	2	2	2	2	2	2	2	2	2	2	2
α	1	1	1	1	1	1	1	2	2	2	2	2	2
β	1	1	2	3	4	5	6	1	2	3	4	5	6
γ	1	1	2	3	51	52	53	3	4	5	53	54	55

TABLE 3: Continuous and discrete parameter forms.

Parameter Type	Continuous Forms	Discrete Forms		
Cartesian coordinates	x_n	x_n^{γ}	$x_n^{\tau\epsilon\alpha}$	$x_n^{\tau\epsilon\alpha\beta}$
Position Vectors	$\vec{r} = \sum_n x_n \vec{e}_n$	$\vec{r}^{\gamma} = \sum_n x_n^{\gamma} \vec{e}_n$	$\vec{r}^{\tau\epsilon\alpha} = \sum_n x_n^{\tau\epsilon\alpha} \vec{e}_n$	$\vec{r}^{\tau\epsilon\alpha\beta} = \sum_n x_n^{\tau\epsilon\alpha\beta} \vec{e}_n$
Displacement Basis Functions	$q_m = q_m(\vec{r})$	$q_m^{\gamma} = q_m(\vec{r}^{\gamma})$	$q_m^{\tau\epsilon\alpha} = q_m(\vec{r}^{\tau\epsilon\alpha})$	$q_m^{\tau\epsilon\alpha\beta} = q_m(\vec{r}^{\tau\epsilon\alpha\beta})$
Assumed Displacements	$u_n = u_n(\vec{r})$	$u_n^{\gamma} = u_n(\vec{r}^{\gamma})$	$u_n^{\tau\epsilon\alpha} = u_n(\vec{r}^{\tau\epsilon\alpha})$	$u_n^{\tau\epsilon\alpha\beta} = u_n(\vec{r}^{\tau\epsilon\alpha\beta})$
Displacement Vectors	$\vec{u} = \sum_n u_n \vec{e}_n$	$\vec{u}^{\gamma} = \sum_n u_n^{\gamma} \vec{e}_n$	$\vec{u}^{\tau\epsilon\alpha} = \sum_n u_n^{\tau\epsilon\alpha} \vec{e}_n$	$\vec{u}^{\tau\epsilon\alpha\beta} = \sum_n u_n^{\tau\epsilon\alpha\beta} \vec{e}_n$

these parameters; and m', n', i' and j' are alternates of their unprimed counterparts. The value of this nomenclature is that it facilitates the uncoupling of global from local dependencies since parameters that are dependent upon α and/or γ are globally dependent; and those that are independent of α and/or γ, but dependent upon τ, ε and/or β, are locally or cluster dependent.

3. The Given Lattice Problem

The lattice cluster stiffness equations are given in the form

$$\overline{F}_n^\gamma = \sum_{n'} \sum_\beta \overline{k}_{nn'}^{\tau\varepsilon\beta} \overline{u}_{n'}^{\tau\varepsilon\alpha\beta} \quad \ldots \quad (\tau,\varepsilon,\alpha) = g^{-1}(\gamma) \qquad 3.1$$

where: the overbar refers to the lattice; $\overline{F}_n^\gamma$ is the component of the external force acting on the lattice at joint γ in the direction of the global unit base vector $\vec{e}_n$; $\overline{k}_{nn'}^{\tau\varepsilon\beta}$ are the cluster stiffness coefficients, or simply the "stiffnesses;" and $\overline{u}_{n'}^{\tau\varepsilon\alpha\beta}$ are the displacements of the joints in the cluster. In the case that the lattice is a truss (i.e., all joints are pinned), equation 3.1 is the exact form of the static equilibrium equation.

If the lattice is a frame (i.e., all joints are rigid) or a hybrid of pinned, rigid and/or flexible joints, the equilibrium equations can be reduced to the form of equation 3.1 by (a) invoking St. Venant's principle to neglect the internal moments at joints β in the cluster which are remote from the principal joint "ε"; and (b) using a standard condensation procedure to eliminate the external moment equations.

We shall assume that for each pair (n,γ) where $\overline{F}_n^\gamma$ is unknown, the displacements are prescribed as

$$\vec{u}_n^\gamma = \vec{U}_n^\gamma \qquad\qquad 3.2$$

where: $\vec{U}_n^\gamma$ are known. The <u>problem</u> is to find $\vec{u}_n^\gamma$ that satisfies equation 3.1 wherever $\vec{F}_n^\gamma$ is known.

4. The Continuum Model

A continuum will be said to be "topologically" equivalent to a lattice if the undeformed material domain of the continuum spans a set of material points, called "images," whose geometrical configuration is identical to the undeformed configuration of joints in the lattice. A continuum that is topologically equivalent to a lattice will also be said to be "kinematically" equivalent to it if all deformed configurations of the lattice's joints and their images in the continuum are identical. As we shall see, the choices of material domain and displacment constraints at non-image points of the continuum will be motivated by factors of simplicity.

A continuum model for a lattice is defined as a continuum that is kinematically equivalent to the lattice. The formulation of the model begins with a choice of material domain that is topologically equivalent to the lattice, and a choice of material or constitutive properties that may be specified independently of the lattice. This choice will also be motivated by factors of simplicity.

Static equilibrium on the surface S of the continuum model is satisfied by

$$T_n(\vec{r}) = d_n(\vec{u}) \qquad\qquad 4.1a$$

where: T_n is the component of the surface traction in the $\vec{e}_n$ direction; and d_n is a scalar differential operator. Static equilibrium in the volume V

enclosed by S is satisfied by

$$B_n(\vec{r}) = D_n(\vec{u}) \qquad\qquad 4.1b$$

where: B_n is the component of the body force in the $\vec{e}_n$ direction; and D_n is a scalar differential operator.

Let S^γ be any subdomain $S^\gamma \subseteq S$ containing only one image $\gamma \epsilon S^\gamma$. Let $\delta W^\gamma(T_n)$ be the variation of work done on the model by all the surface tractions T_n acting on S^γ. Then

$$\delta W^\gamma(T_n) = \int\!\!\int_{S^\gamma} d_n(\vec{u})\; \delta u_n\; dS \;. \qquad\qquad 4.2a$$

Similarly,

$$\delta W^\gamma(B_n) = \int\!\!\int\!\!\int_{V^\gamma} D_n(\vec{u}) \delta u_n dV \qquad\qquad 4.2b$$

where: $\delta W^\gamma(B_n)$ and V^γ are work and volume domain analogues of $\delta W^\gamma(T_n)$ and S^γ respectively. These conditions must clearly be satisfied no matter how the loads on the continuum are applied.

We shall stipulate that the loads shall be applied so as to satisfy

$$\delta W^\gamma(T_n) = F_n^\gamma\; \delta u_n^\gamma \;\ldots\; \forall \gamma \epsilon S \;, \text{ and} \qquad\qquad 4.3a$$

$$\delta W^\gamma(B_n) = F_n^\gamma\; \delta u_n^\gamma \;\ldots\; \forall \gamma \epsilon V \qquad\qquad 4.3b$$

for all pairs (n,γ) where $\overline{F}_n^\gamma$ are prescribed, and where:

$$F_n^\gamma = \sum_{n'}\sum_\beta k_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta} + \overline{F}_n^\gamma - \sum_{n'}\sum_\beta \overline{k}_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta},\qquad 4.4a$$

and

$$\sum_{n'}\sum_\beta k_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta}\,\delta u_n^\gamma \cong \begin{cases} \displaystyle\iint_{S^\gamma} d_n(\vec{u})\,\delta u_n\,dS & \dots\ \gamma\in S \\[2em] \displaystyle\iiint_{V^\gamma} D_n(\vec{u})\,\delta u_n\,dV & \dots\ \gamma\in V. \end{cases}\qquad 4.4b$$

The coefficients $k_{nn'}^{\tau\epsilon\beta}$ can be obtained from finite difference approximations of the right-hand sides of equations 4.2a and b. Below, these stiffness coefficients will be obtained from use of the cluster as a device for implementing the finite element method (FEM) on a local scale.

In either case, substitution of equation 4.4a into equations 4.3a and b, and appeal to equations 4.2a and b, give

$$\iint_{S^\gamma} d_n(\vec{u})\delta u_n dS = \left(\sum_{n'}\sum_\beta k_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta} + \overline{F}_n^\gamma - \sum_{n'}\sum_\beta \overline{k}_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta}\right)\delta u_n^\gamma \qquad 4.5a$$

where $\gamma\in S$, and

$$\iiint_{V^\gamma} D_n(\vec{u})\delta u_n dV = \left(\sum_{n'}\sum_\beta k_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta} + \overline{F}_n^\gamma - \sum_{n'}\sum_\beta \overline{k}_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta}\right)\delta u_n^\gamma \qquad 4.5b$$

where $\gamma\in V$. Then, in view of equation 4.4b, we can deduce from equations 4.5a

and b that

$$\overline{F}_n^\gamma - \sum_{n'} \sum_\beta \overline{k}_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\beta} \cong 0 \qquad\qquad 4.6$$

for all pairs (n,γ) where δu_n^γ is arbitrary (i.e., where $\overline{F}_n^\gamma$ are prescribed). By further stipulating that

$$u_n^\gamma = \overline{U}_n^\gamma \qquad\qquad 4.7$$

for all pairs (n,γ) where $\overline{U}_n^\gamma$ are prescribed (i.e., where $\delta u_n^\gamma = 0$), we can conclude that

$$u_n^\gamma = \overline{u}_n^\gamma \quad \ldots \quad \forall (n,\gamma) \qquad\qquad 4.8$$

Thus, the continuum is kinematically equivalent to the lattice.

As long as the axiom of continuity is not violated (i.e., the Jacobian $\neq 0$), kinematic equivalence is preserved regardless of any additional displacement constraints imposed at non-image points in the continuum. Here again simplicity will be seen as the motivating factor for imposing displacement constraints on the continuum at non-image points.

Let the "load correction" be defined as the force

$$\Delta F_n^\gamma = \sum_{n'} \sum_\beta \Delta k_{nn'}^{\tau\epsilon\beta} u_{n'}^{\tau\epsilon\alpha\beta} \qquad\qquad 4.9a$$

where

$$\Delta k^{\tau \epsilon \beta}_{nn'} = k^{\tau \epsilon \beta}_{nn'} - \overline{k}^{\tau \epsilon \beta}_{nn'} . \qquad 4.9b$$

Then

$$F^{\gamma}_n = \overline{F}^{\gamma}_n + \Delta F^{\gamma}_n . \qquad 4.10$$

In summary:

A continuum that is topologically equivalent to a lattice is also kinematically equivalent to it if (1) the displacement prescriptions imposed on the lattice are imposed on the continuum at their corresponding images; and (2) the load on the continuum is the sum of the lattice forces and the load corrections.

This principle, called the "load correction principle," is the strategy by which the given lattice problem was mapped into a continuum model.

5. Solution of the Model

An approximate global solution of the continuum model will be obtained using a Ritz approach. The basis functions for the assumed displacement field are

$$q_m = q_m(\vec{r}) \qquad 5.1$$

where: the range of m is finite; and the various discrete forms q^{γ}_m, $q^{\tau \epsilon \beta}_m$

and $q_m^{\tau\epsilon\alpha\beta}$ are given in Table 3. To simplify the analysis, two special conditions will be imposed on this set of basis functions:

$$q_m^{\tau\epsilon\alpha\beta} = \sum_{m'} J_{mm'}^{\tau\epsilon\beta} \, q_{m'}^{\tau\epsilon\alpha} \text{ , and} \qquad\qquad 5.2a$$

$$\partial q_m/\partial x_n = \sum_{m'} D_{nmm'} q_{m'} \text{ ,} \qquad\qquad 5.2b$$

where: $J_{mm'}^{\tau\epsilon\beta}$ and $D_{nmm'}$ are known constants.

This set if basis functions will generate the assumed displacement field in the form

$$u_n = \sum_{m} b_{nm} q_m + \sum_{m} \sum_{j} a_{nmj} \, q_m \, \lambda_j \qquad\qquad 5.3$$

where: u_n is defined in Table 3; and b_{nm} and a_{nmj} are coefficients that are specified so that the displacement boundary conditions are satisfied for all choices of the varying parameters λ_j. Then, the various discrete forms of u_n shown in Table 3 can be expressed in terms of λ_j. Furthermore, the variation

$$\delta u_n = \sum_{m} \sum_{j} a_{nmj} q_m \delta\lambda_j \qquad\qquad 5.4$$

and its various discrete forms follow as suggested by Table 3.

Using these equations, the forces applied to the continuum can be written as

$$F_n^\gamma = \overline{F}_n^\gamma + \sum_m b_{nm}^{**\tau\epsilon} q_m^{\tau\epsilon\beta} + \sum_m \sum_j a_{nmj}^{**\tau\epsilon} q_m^{\tau\epsilon\alpha} \lambda_j \qquad 5.5$$

where: the constants $b_{nm}^{**\tau\epsilon}$ and $a_{nmj}^{**\tau\epsilon}$ are given in Appendix 2. The variation of work on the model, then, is

$$\delta W = \sum_\gamma F_n^\gamma \, \delta u_n^\gamma \qquad 5.6a$$

which, after substitutions from equations 5.5 and 5.3, becomes

$$\delta W = \sum_j (\phi_j^{(F)} + \phi_j^{(b)} + \sum_{j'} \phi_{jj'}^{(a)} \lambda_{j'}) \delta \lambda_j \qquad 5.6b$$

where: $\phi_j^{(F)}$, $\phi_j^{(b)}$, and $\phi_{jj'}^{(a)}$ are defined in Appendix 2.

Using engineering notation, the components of strain in the continuum are

$$\epsilon_i = \sum_n \sum_{n'} G_{inn'} \frac{\partial u_n}{\partial x_{n'}} \qquad 5.7$$

in the linear case where: $G_{inn'}$ are known constants. Differentiation of equation 5.3, and substitution of equation 5.2b into the result yield

$$\epsilon_i = \sum_m (G_{im}^{(b)} + \sum_j G_{imj}^{(a)} \lambda_j) q_m \qquad 5.8$$

where: $G_{im}^{(b)}$ and $G_{imj}^{(a)}$ are given in Appendix 2.

If we assume that the model obeys Hooke's law, the stress can be written as

$$\sigma_i = \sum_{i'} C_{ii'} \epsilon_{i'} \qquad\qquad 5.9$$

where: $C_{ii'}$ are prescribed constitutive constants. Then, the variation of internal energy

$$\delta P = \sum_i \iiint_V \sigma_i \, \delta\epsilon_i \, dV \qquad\qquad 5.10$$

can be written as

$$\delta P = \sum_j (\Psi_j^{(b)} + \sum_{j'} \Psi_{jj'}^{(a)} \lambda_{j'}) \delta\lambda_j \qquad\qquad 5.11$$

after appropriate substitutions of equations 5.8 and 5.9 into equation 5.10, and where: $\Psi_j^{(b)}$ and $\Psi_{jj'}^{(a)}$ are given in Appendix 2.

Since $\delta P = \delta W$, and since $\delta\lambda_j$ are arbitrary, the solution for λ_j proceeds from (a) setting equation 5.6b equal to equation 5.11 to get

$$\sum_{j'} (\Psi_{jj'}^{(a)} - \phi_{jj'}^{(a)}) \lambda_{j'} = \Psi_j^{(b)} - \phi_j^{(b)} - \phi_j^{(F)} ; \qquad\qquad 5.12$$

and (b) implementing the standard Gaussian elimination procedure. Back substitution of the resulting λ_j into equation 5.3 gives the continuous displacement function for the model, and the discrete form u_m^γ is the solution of the lattice.

6. Computational Economy

The effort required to obtain an FEM solution to the given lattice problem

can be measured roughly in terms of the cpu (and memory) required to solve a narrowly banded set of algebraic equations of order ΓN where: $1 < \gamma < \Gamma$; and $1 < n < N$. The effort required to obtain and LCM solution to the same problem can be measured roughly in terms of the cpu required to (a) solve a dense set of algebraic equations of order J, where $1 < j < J$ (see equation 5.12); and to (b) evaluate the parameters listed in the Appendix. As we shall see, $J << \Gamma N$ generally, and the evaluation of these parameters can be made with an effort that is generally independent of the ranges of the global indexes α and γ.

From Appendix 2, it is clear that if the computational effort required to evaluate the parameters ϕ_{nm} and $Q_{nm}^{\tau\epsilon}$, is independent of A (where $1 < \alpha < A$) and Γ, then the effort required to evaluate the remaining parameters is also independent of A and Γ. To see what effort is actually required to evaluate these parameters, consider the equation

$$\phi_{nm} = \sum_{\gamma} \overline{F}_n^\gamma q_m^\gamma \; .$$
6.1

An important role of the LCM in the analysis of large space structures is to produce a generalized or reduced-order flexibility matrix for the lattice so that the frequencies and mode shapes can be deduced from

$$[I - \omega^2 fM]\{\mu\} = 0$$
6.2

where: I is an identity matrix; ω is a natural frequency; f is the flexibility matrix, which is the inverse of the stiffness matrix; M is the generalized mass matrix; and μ is the mode shape vector. The flexibility matrix will be generated column-wise from sequential applications of a single unit force to the model. Thus, each use of equation 6.1 will usually involve only

one non-zero value of $\overline{F}_n^\gamma$. This is to say that

$$\phi_{nm} = \overline{F}_{n_o}^\gamma \, q_{m_o}^\gamma \, \delta_{n_o n} \delta_{m_o m} \cdots \text{ no sum on } n_o \text{ and } m_o \;, \qquad 6.3$$

where: the indexes n_o and m_o are fixed values; and $\delta_{n_o n}$ and $\delta_{m_o m}$ are Kronecker deltas.

If gravity gradients, solar pressure or other distributed loads are applied to the lattice, then equation 6.1 can be computed economically if we can write

$$\overline{F}_n^\gamma = \sum_m c_{nm} q_m^\gamma \qquad 6.4$$

where: c_{nm} are known constants. Substitution of this form into equation 6.1 yields

$$\phi_{nm} = \sum_{m'} c_{nm'} \left(\sum_\gamma q_m^\gamma q_{m'}^\gamma \right) \;, \qquad 6.5$$

The term in parentheses can be evaluated efficiently using a Dirac-delta technique.

For example, if

$$q_2^\gamma \, q_2^\gamma = (x_1^\gamma)^2 \qquad 6.6$$

then

$$\sum_\gamma q_2^\gamma q_2^\gamma = \sum_\gamma (x_1^\gamma)^2 \qquad 6.7$$

But, using the Dirac-delta function, we can write

$$(x_1^\gamma)^2 = \int_{-\infty}^{+\infty} \xi^2 \, \delta(\xi - x_1^\gamma) d\xi \; , \tag{6.8}$$

so that

$$\sum_\gamma q_2^\gamma q_2^\gamma = \int_{+x_1^1 - 0}^{+x_1^\Gamma + 0} \xi^2 \left[\sum_\gamma \delta(\xi - x_1^\gamma)\right] d\xi \tag{6.9}$$

where the term in brackets is an integrable function. Integrating equation 6.9 by parts gives

$$\sum_\gamma q_2^\gamma q_2^\gamma = [\xi^2 Z_1 - \xi Z_2 + Z_3]_{x_1^1}^{x_1^\Gamma} \tag{6.10}$$

where: $Z_1 = \int \sum_\gamma \delta(\xi - x_1^\gamma) d\xi$; and $Z_{k+1} = \int Z_k d\xi$, $k=1,2$. The integrals Z_1, Z_2, Z_3 can be integrated exactly – by hand. Back substitution of these values into equation 6.10, and then into equation 6.5 results in a symbolic evaluation of ϕ_{nm}. Thus, the cpu required to evaluate ϕ_{nm} is independent of Γ, and the memory requirements are virtually miniscule.

In a like manner, the parameter

$$Q_{mm'}^{\tau\varepsilon} = \sum_\alpha q_m^{\tau\varepsilon\alpha} q_{m'}^{\tau\varepsilon\alpha} \tag{6.11}$$

is evaluated using the above Dirac-delta technique, thereby implementing a globally independent effort. Thus, the LCM can be implemented economically.

7. Sample Problems

A selection of plane lattice structures is discussed below as a means of demonstrating the versatility of the LCM. In each case, the continuum model is an isotropic, plane stress solid whose material domain follows the outline of the lattice. Where the displacement boundary conditions for the lattice are discrete hinge supports, the model is constrained to obey hinge support conditions along line segments joining the images of these hinges. The assumed displacement field is generated by power basis functions

$$\{q_m: 1, x_1, x_2, (x_1)^2, x_1 x_2, (x_2)^2, (x_1)^3, (x_1)^2 x_2, \ldots \}. \quad 7.1$$

The LCM was implemented via the LCM21 computer code on a Vax 11/750[4], and the FEM was implemented via STRUDL on an IBM 3033[5].

Problem #1. Two loading cases are considered for the cantilever truss shown in Figure 3 . The truss is a single-braced, one-way, single-story structure that is well known to behave much like an orthotropic beam. The constitutive properties, however, will be modeled as isotropic in the following LCM model.

In the boom's case , the LCM models the horizontal displacement field with a linear displacement field. This model is sufficient no matter what the length of the boom may be. Thus, the cpu required by the LCM is independent of the global d.o.f. of the boom (see Problem #2 below).

In the bending case, the LCM models the horizontal displacement field with a parabola, and the vertical with cubic parabola. Comparison with the FEM is adequate, and cpu remains independent of the global d.o.f. of the system.

Problem #2. The boom-like loading case is considered for the cantilever lattice shown in Figure 4 . Two cases are considered: truss (or pinned jointed); and frame (or rigid jointed).
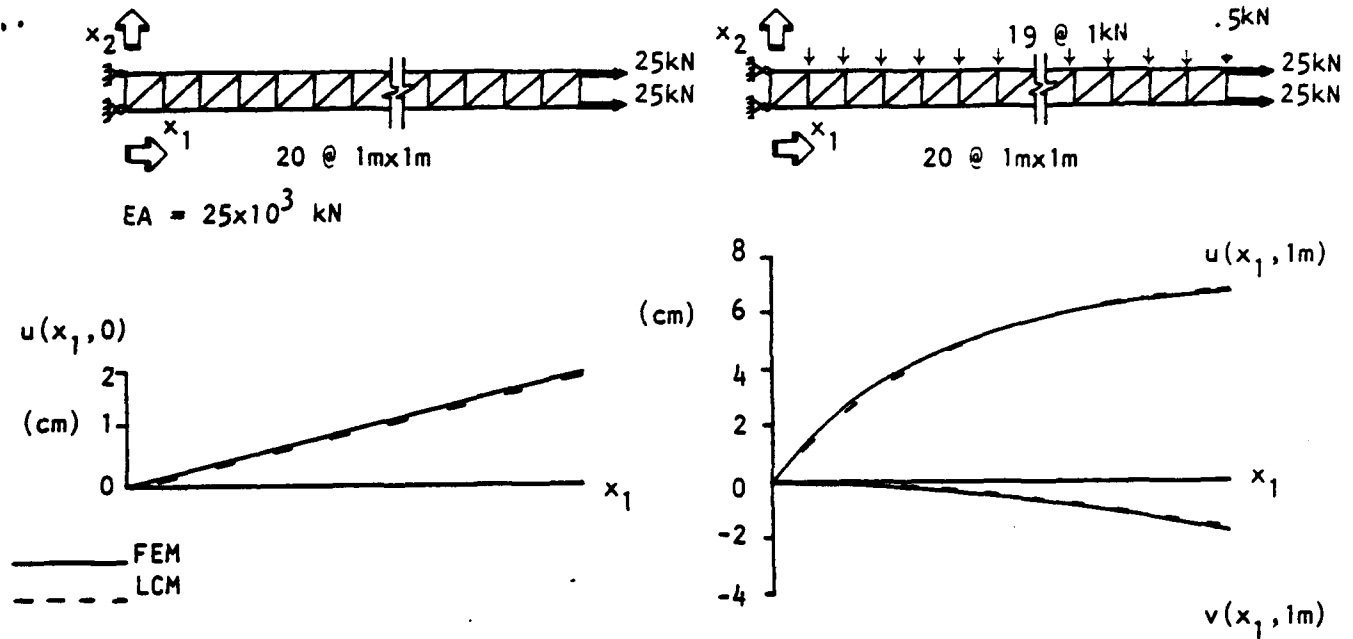
x_2 ⬆ 25kN
25kN
⬆x_1 20 @ 1mx1m

EA = 25x10³ kN

x_2 ⬆   19 @ 1kN   .5kN
25kN
25kN
⬆x_1 20 @ 1mx1m

u(x_1,0)

(cm)

FEM
LCM

(cm)   u(x_1,1m)

x_1

v(x_1,1m)

FIGURE 3: Displacements of two singly-braced booms.

x_2 ⬆   5.6 kips
5.6 kips
⬆x_1 20 @ 3'x3'

(10⁻³ in)

Frame

Truss

v(x_1,1m)

u(x_1,0)

(in)

Truss

Frame

x_1

FEM   EA = 5.62x10³ kips
LCM   EI = 2.12x10³ kip-ft

cpu(secs)

Frame
Truss
Frame
Truss

Bays
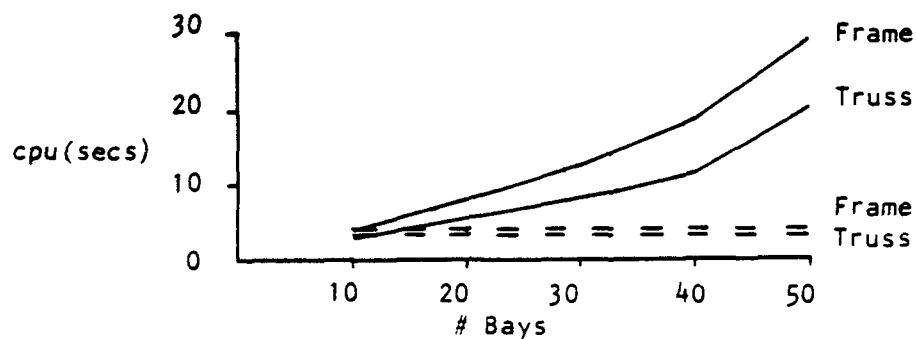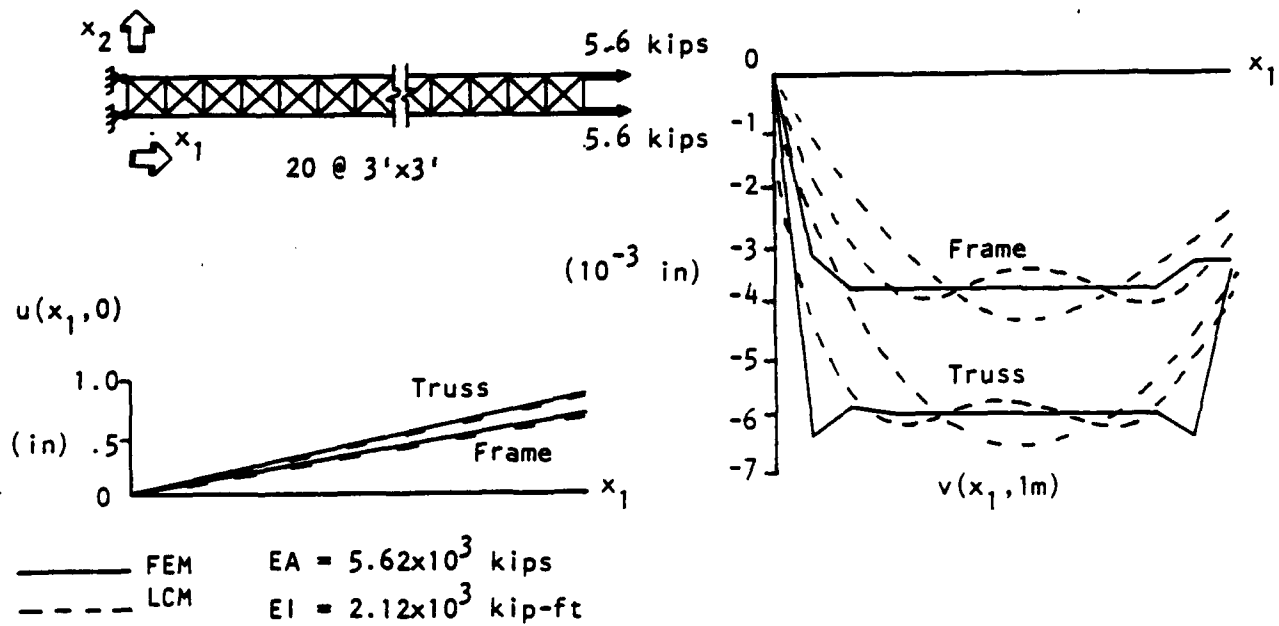
FIGURE 4: Pin-jointed (truss) and rigid-jointed (frame) doubly-
braced booms.

A linear displacement field is used to model the horizontal displacements, and various orders of parabolic fields are considered for the vertical displacements. A cluster of two panels was sufficient for the truss, and five panels were used in the case of the frame. While increasingly larger displacement fields for v are required to capture the edge effects in increasingly longer booms, a linear displacement field remains sufficient for u whatever the overall length of the boom. Because the d.o.f. of these displacement fields are small, the LCM cpu is <u>virtually</u> independent of the length (i.e., # bays), while the FEM cpu increases geometrically.

<u>Problem #3</u>. The boom-like complex case is considered for the doubly-braced hybrid of pinned and rigid joints shown in Figure 5 . The horizontal displacement field is linear, and the vertical is of 10 d.o.f.

The LCM and FEM results for the horizontal displacements are almost indistinguishable. The vertical displacements, however, compare favorably in magnitude, sign, and overall trends. Edge effects are captured to the degree allowed by the assumed displacement field (Figure 5).

<u>Problem #4</u>. Shown in Figures 6-8 are two-story cantilever lattices[6]. Three local architectural patterns are considered: doubly-braced, diamond pattern of alternating diagonals, and W-pattern of alternating diagonals. Comparisons of FEM with LCM results are shown, and these comparisons generally follow those observed in the single-story cases: longitudinal displacements for booms approximated well in the linear case; bending results were good; etc. There were, however, some exceptions.

The bending cases, for relatively short beams, required higher-order approximations for the displacement fields, and cpu for the short beams com-
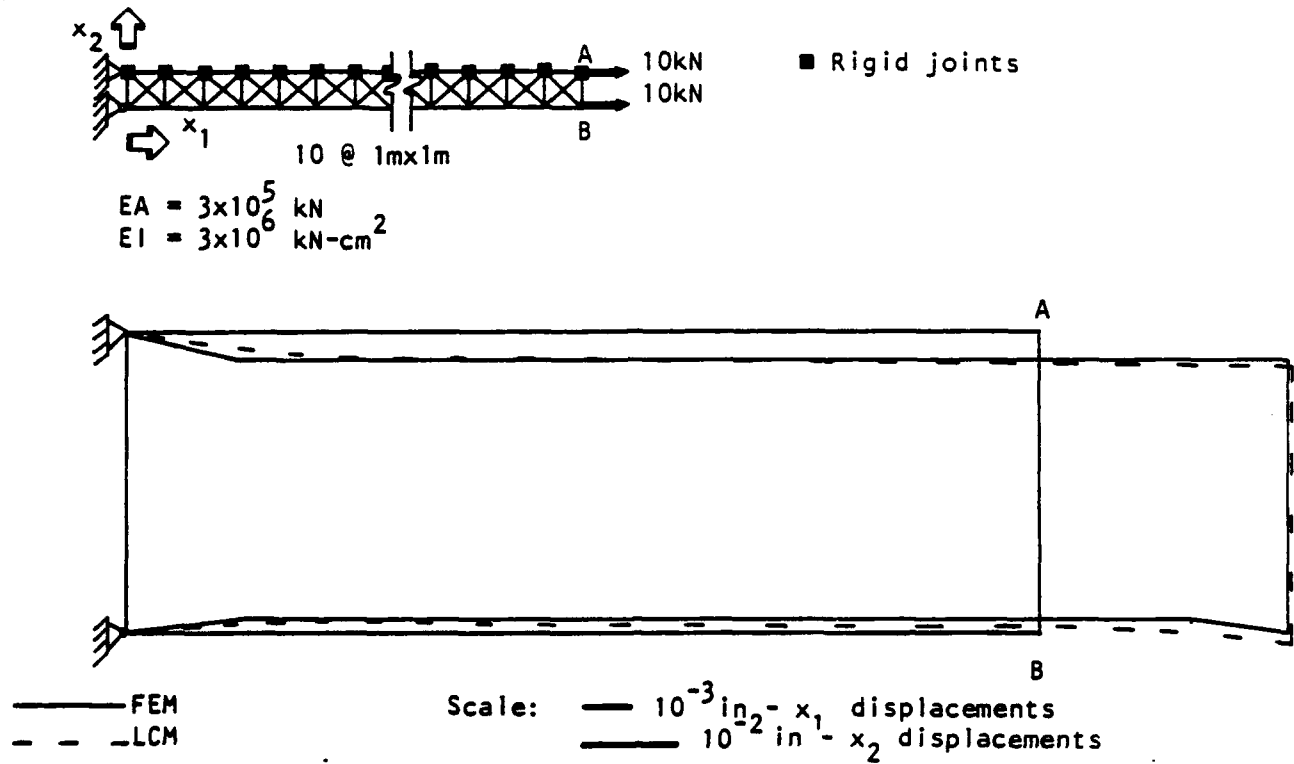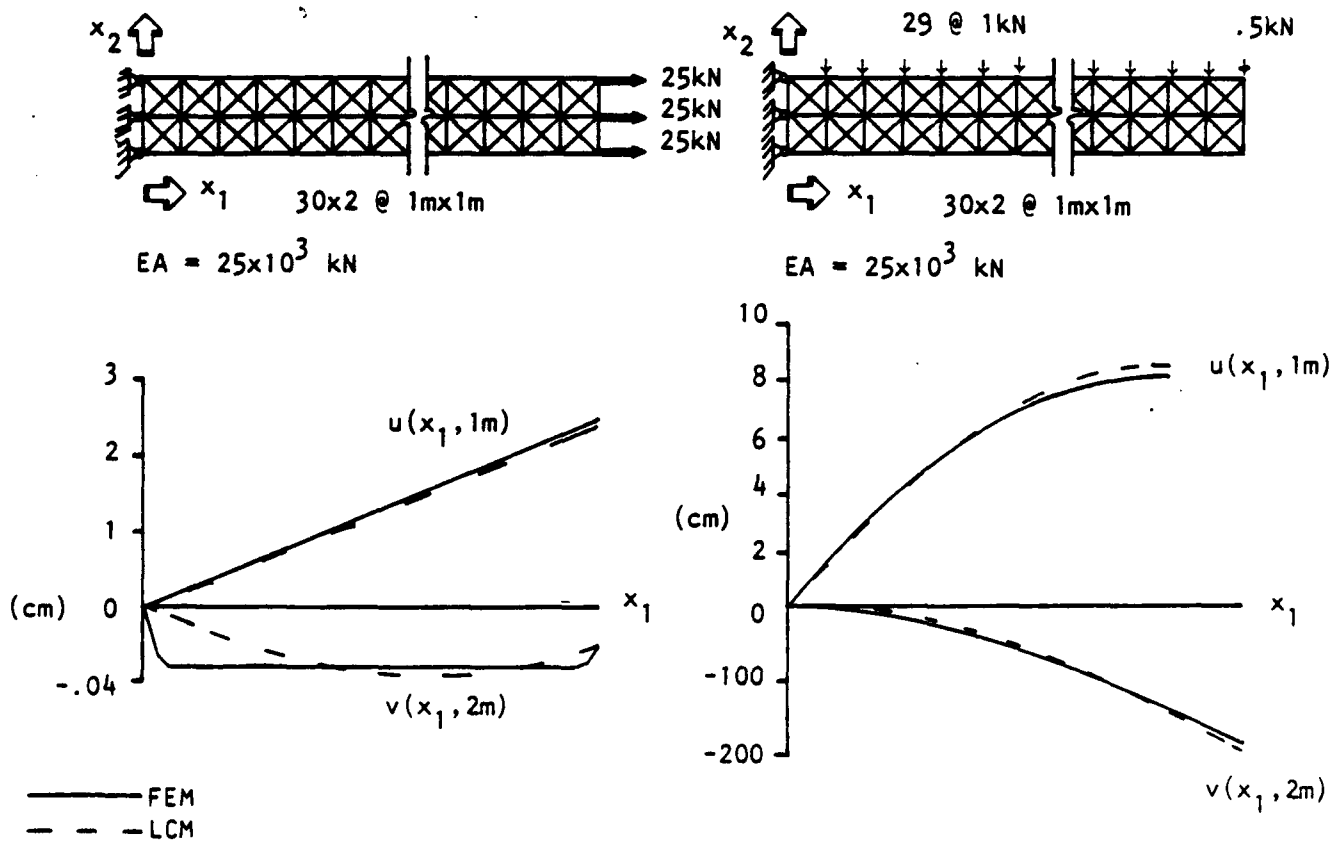
FIGURE 5: Hybrid truss/frame boom.

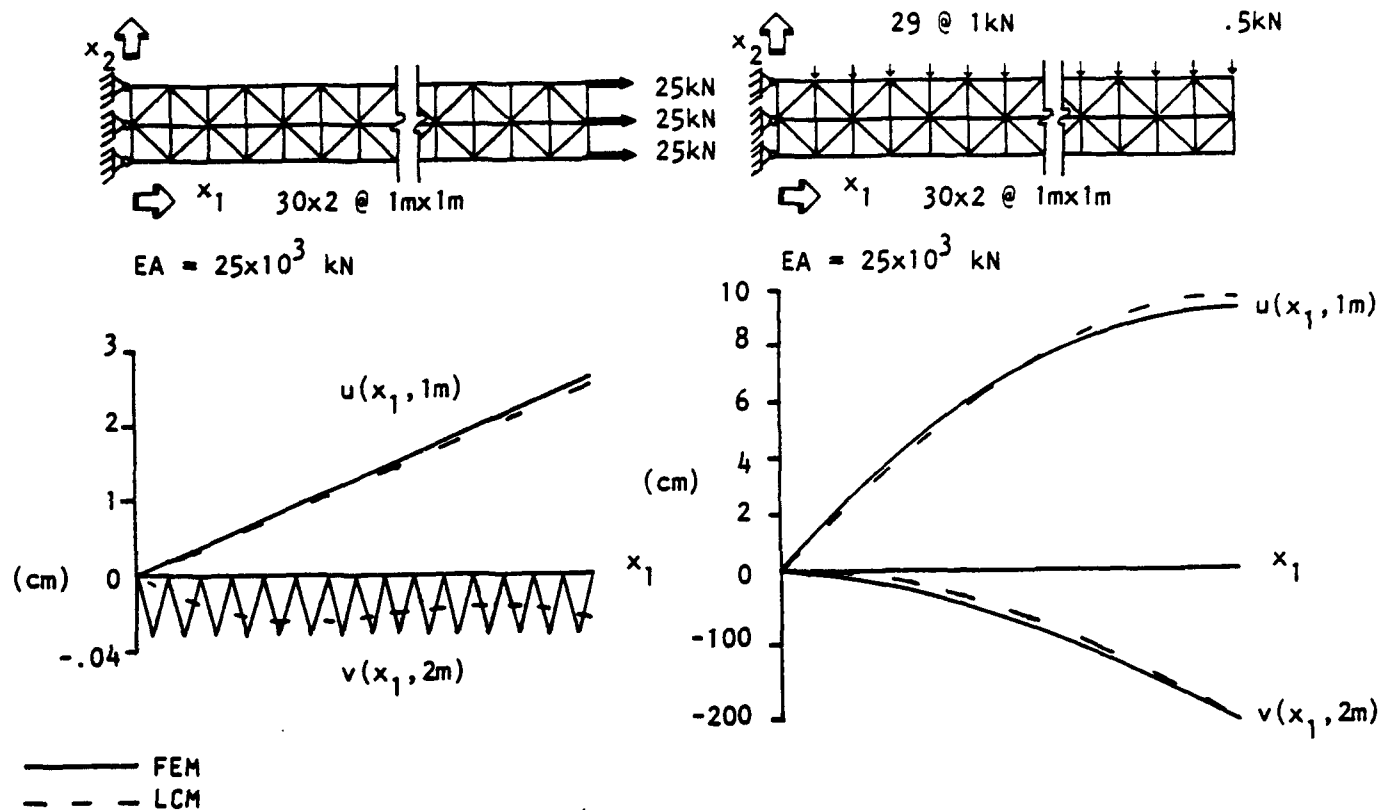FIGURE 6: Two-story, doubly-braced cantilever trusses.
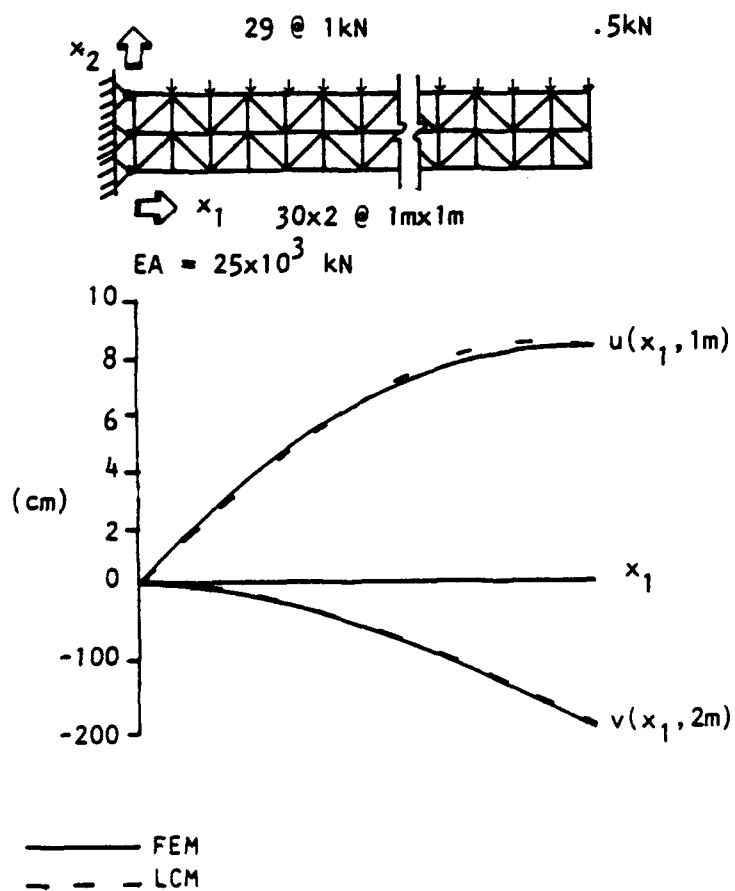
FIGURE 7: Two-story, diamond-braced cantilever trusses.

FIGURE 8: Two-story, W-braced
cantilever truss.

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pared with FEM cpu. However, FEM cpu was not competitive with LCM cpu for longer beams.

The vertical displacements shown in Figure 7 were not approximated well. Although horizontal displacements were good in this case, as much as 12% error in these displacements was observed in the single-story case. However, at the expense of non-competitive cpu, the LCM accurately predicted the displacement behavior of short booms assembled from this pattern in both the single-story and the two-story cases[7]. The failure of the LCM in the longer cases is, therefore, attributable to the choice of assumed basis functions of the power type where the actual vertical displacement field varies periodically in the same period as the local repeating architectures (see Figure 7). A Fourier basis set may well resolve this problem.

Problem #5. The flat, in-plane deforming plates shown in Figure 9 are considered. Singly-braced, alternating diagonal trusses are considered, and a hybrid LCM/FEM analysis is compared with a conventional global FEM analysis.

The problem is to compute the vertical displacement under the load applied to the 5-bay x 5-bay plate. Two convergent cases are considered. First , the FEM is applied to increasingly larger subdomains of the plate assuming the internal joints are pinned.

Second, the LCM is used to compute the displacements of the 5-bay x 5-bay plate. These results are then used as boundary conditions at the internal joints where the FEM. is again applied.

The second case converges much faster than the first, producing an 18% improvement in system order at the 15 d.o.f. subdomain. Thus, the hybrid LCM/FEM is more economical than the conventional global FEM for this case.
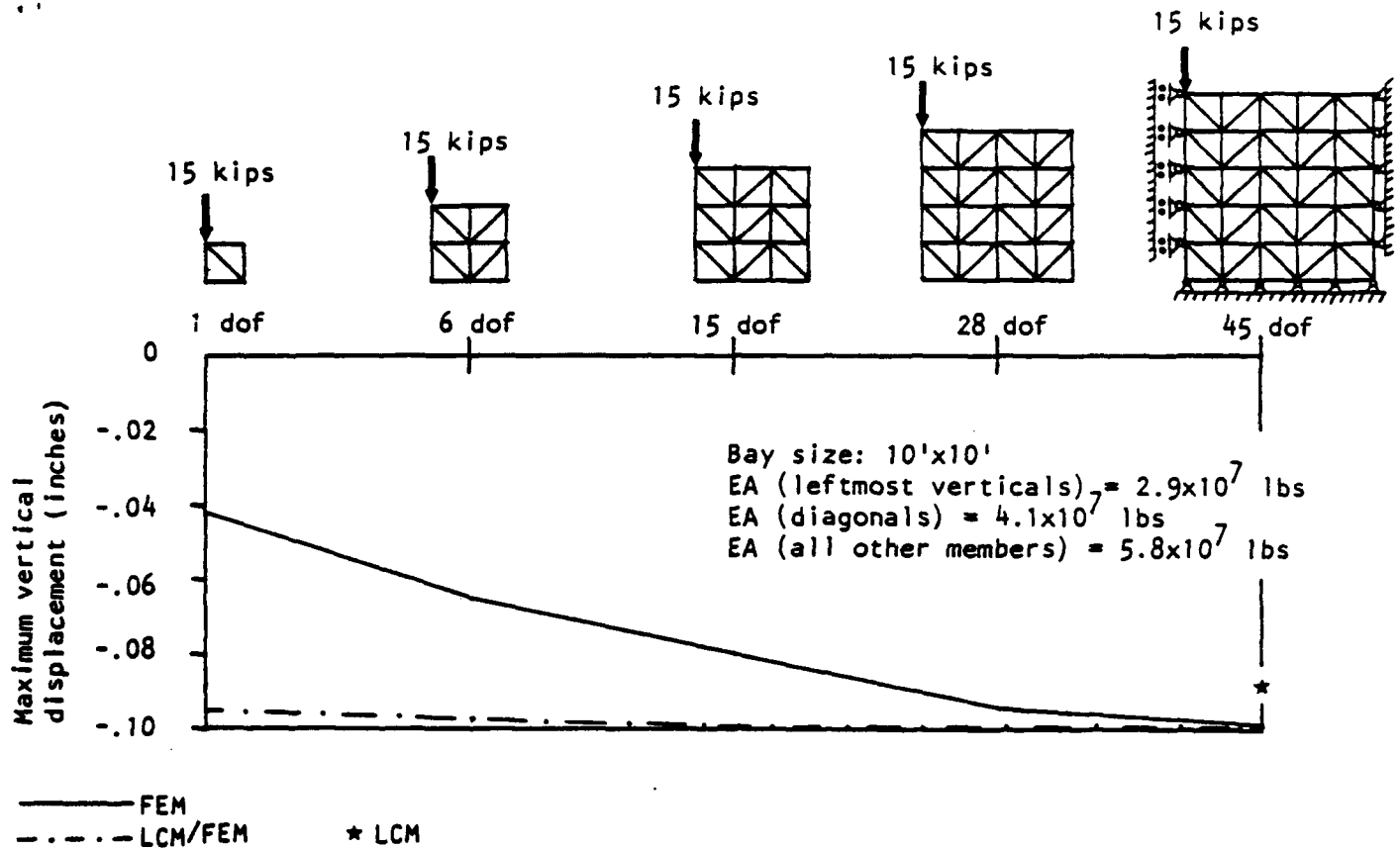
FIGURE 9: In-plane-deforming plate-like lattice.

8. Summary and Conclusions

A nomenclature was introduced in section 2 that facilitated the uncoupling of cluster dependent parameters from globally dependent ones. The given lattice problem was then formulated in terms of cluster coordinates. A continuum model was formed as a continuum that is topologically and kinematically equivalent to the lattice, but subject to a displacement-dependent loading called the "load correction." Using this load correction, the continuum model was described in terms of cluster coordinates.

The model was solved using a Ritz procedure which introduced globally dependent parameters. However, when considering lattices with repeating cluster architectures - as will typify LSS - these parameters were reduced to cluster dependencies using a Dirac-delta technique for computing long summations. Sample problems were presented which demonstrated that the range of lattice types to which the LCM is adaptable exceeds the present scope of competing continuum modeling methods. While these methods lead to closed-form solutions, the LCM leads to numerical models.

Some important features of the LCM are:

- General applicability to pin-jointed, rigid-jointed, and hybrids of pinned- and rigid-jointed lattices;

- Micropolar models are not necessary for rigid-jointed lattices as the cluster is a means of invoking St. Venant's principle locally so that the rotational degrees of freedom can be eliminated via conventional static condensation processes;

- The matrices comprising the LCM are either the familiar FEM stiffness matrices whose dimensions are of the cluster d.o.f., or are otherwise

easily generated;

- The LCM can be applied to a wide range of lattices possessing different local and global architectures without adjusting the methodology to accommodate them;

- The LCM accommodates a choice of assumed displacement fields. Although present considerations included power basis functions, Fourier basis function may be desirable in cases where the displacements of the truss vary periodically in the same period as the clusters;

- The LCM has the ability to capture edge effects;

- The global response of the LCM can be captured at a cpu that is substantially less than that of FEM for large d.o.f. structures. In most cases observed, the LCM cpu is virtually independent of the d.o.f.

- Hybrid LCM/FEM promises to serve as a comprehensive analysis tool for LSS that is superior to global implementation of the FEM.

Finally, well-known generalized or lumped mass techniques provide the basis for modal analysis of LSS. The LCM or hybrid LCM/FEM can be used as an economical means of obtaining the generalized flexibility matrices for LSS. Thus, a reduced basis dynamic analysis technique for LSS proceeds from use of the LCM as a static tool.

APPENDIX 1 : NOMENCLATURE

a_{nmj}, $a_{nmj}^{*\tau\epsilon\beta}$, $a_{nmj}^{**\tau\epsilon}$	Parameters
B_n	Body force components
b_{nm}, $b_{nm}^{*\tau\epsilon\beta}$, $b_{nm}^{**\tau\epsilon}$	Parameters
$C_{ii'}$	Constitutive coefficients
c_{nm}	Coefficients
$D_{nmm'}$	Coefficients
$\mathbf{D}_n$, $\mathbf{d}_n$	Differential operators
$\vec{e}_n$	Cartesian unit base vectors
$\overline{F}_n^{\gamma}$, F_n^{γ}, ΔF_n^{γ}	Forces
f	Flexibility matrix
$\mathbf{G}$, $\mathbf{g}$	Index operators
$G_{inn'}$, $G_{im}^{(b)}$, $G_{imj}^{(a)}$	Parameters
I	Identity matrix
i	Index
$J_{nm}^{\tau\epsilon\beta}$	Parameter
j, J	Indexes, $1 < j < J$
$\overline{k}_{nn'}^{\tau\epsilon\beta}$, $\kappa_{nn'}^{\tau\epsilon\beta}$, $\Delta k_{nn'}^{\tau\epsilon\beta}$	Stiffness
M	Mass matrix
m, m_o	Indexes
N	Index
n, n_o	Indexes
P	Internal energy
$Q_{mm'}$, $Q_{mm'}^{\tau\epsilon}$, $Q_{nm}^{(a)}$, $Q_{nm}^{(b)}$	Parameters

q_m, q_m^{γ}, $q_m^{\tau\epsilon\beta}$, $q_m^{\tau\epsilon\alpha\beta}$	Displacement basis functions
$\vec{r}$	Position vector
S, S^{γ}	Surface areas
T_n	Surface traction
$\vec{u}$, u_n, u_n^{γ}, $u_n^{\tau\epsilon\alpha}$, $u_n^{\tau\epsilon\alpha\beta}$, $\vec{U}_n^{\gamma}$	Displacements
V, V^{γ}	Volumes
W, $W^{\gamma}(T_n)$, $W^{\gamma}(B_n)$	Work
x_n, x_n^{γ}, $x_n^{\tau\epsilon\alpha}$, $x_n^{\tau\epsilon\alpha\beta}$	Cartesian coordinates
Z_1, Z_2, Z_3	Integrals of the Dirac-delta function
α, A	Indexes $1 < \alpha < A$
β	Index
γ, Γ	Indexes, $1 < \gamma < \Gamma$
δ	Operator: variational, Dirac-delta
δ_{nm}	Kronecker delta
ϵ	Index
ϵ_i	Strain
λ_j	Varying coefficients
μ	Mode shape vector
ξ	Dummy integration variable
σ_i	Stress
τ	Index
$\phi_i^{(F)}$, $\phi_i^{(b)}$, $\phi_{jj'}^{(a)}$, ϕ_{nm}	Parameters
$\psi_j^{(b)}$, $\psi_{jj'}^{(a)}$	Parameters
ω	Frequency

APPENDIX 2 : MAIN PARAMETER DEFINITIONS

1. $a_{nmj}^{*\tau\epsilon\beta} = \sum\limits_{m'} a_{nm'j} \, J_{m'm}^{\tau\epsilon\beta}$

2. $b_{nm}^{*\tau\epsilon\beta} = \sum\limits_{m'} b_{nm'} \, J_{m'm}^{\tau\epsilon\beta}$

3. $a_{nmj}^{**\tau\epsilon} = \sum\limits_{n'} \sum\limits_{\beta} \Delta k_{nn'}^{\tau\epsilon\beta} \, a_{n'mj}^{*\tau\epsilon\beta}$

4. $b_{nm}^{**\tau\epsilon} = \sum\limits_{n'} \sum\limits_{\beta} \Delta k_{nn'}^{\tau\epsilon\beta} \, b_{n'm}^{*\tau\epsilon\beta}$

5. $\phi_{nm} = \sum\limits_{\gamma} \overline{F}_n^{\gamma} \, q_m^{\gamma}$

6. $\phi_j^{(F)} = \sum\limits_{n} \sum\limits_{m} \phi_{nm} \, a_{nmj}$

7. $Q_{mm'}^{\tau\epsilon} = \sum\limits_{\alpha} q_m^{\tau\epsilon\alpha} \, q_{m'}^{\tau\epsilon\alpha}$

8. $Q_{nm}^{(b)} = \sum\limits_{m'} \sum\limits_{\tau} \sum\limits_{\epsilon} b_{nm'}^{**\tau\epsilon} \, Q_{m'm}^{\tau\epsilon}$

9. $\phi_j^{(b)} = \sum\limits_{n} \sum\limits_{m} Q_{nm}^{(b)} \, a_{nmj}$

10. $Q_{nmj}^{(a)} = \sum\limits_{m'} \sum\limits_{\tau} \sum\limits_{\epsilon} a_{nm'j}^{**\tau\epsilon} \, Q_{m'm}^{\tau\epsilon}$

11. $\phi_{jj'}^{(a)} = \sum\limits_{n} \sum\limits_{m} Q_{nmj'}^{(a)} \, a_{nmj}$

12. $G_{im}^{(b)} = \sum_n \sum_{n'} \sum_{m'} G_{inn'} b_{nm'} D_{n'm'm}$

13. $G_{imj}^{(a)} = \sum_n \sum_{n'} \sum_{m'} G_{in'n} a_{n'm'j} D_{nm'm}$

14. $Q_{mm'} = \iiint\limits_V q_m q_{m'} \, dV$

15. $v_j^{(b)} = \sum_i \sum_{i'} \sum_m \sum_{m'} C_{ii'} G_{i'm'}^{(b)} G_{imj}^{(a)} Q_{mm'}$

16. $v_{jj'}^{(a)} = \sum_i \sum_{i'} \sum_m \sum_{m'} C_{ii'} G_{i'm'j}^{(a)} G_{imj'}^{(a)} Q_{mm'}$

APPENDIX 3 : REFERENCES

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2. See, for example, Noor, A.K., Anderson, N.S. and Greene, W.H., "Continuum Mechanics for Beam- and Platelike Lattice Structures," AIAA Journal, Vol. 16, Dec., 1978, pp. 1219-1228; or Renton, J.D., "The Beam-like Behavior of Space Trusses," AIAA Journal, Vol. 22, Feb. 1984, pp. 273-280.

3. Broome, T.H. and Stubbs, N., "The Load Correction Principle," presented at the SES Annual Meeting, SUNY Buffalo, Buffalo, NY, August 27, 1986.

4. Fortran version of LCM21 resides on a VAX 11/750 obtained from DoD grant #AFOSR-84-0169.

5. This version of STRUDL resides on the IBM 3033 at Howard University.

6. McKissack, C., "Versatility of the Load Correction Method," M.S. Thesis, Howard University, Fall 1986.

7. Ibid.

7. COMPUTATIONAL ISSUES IN THE APPLICATION

OF THE LCM TO MESOMECHANICS

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ABSTRACT

This paper addresses computational issues associated with a function D: $I \times I \longrightarrow R$ defined by the relation

$$D(N,I) = h^{1-I} \sum_{n=1}^{N} \int_{x_1-0}^{x_N+0} \int_{x_1-0}^{z_I} \cdots \int_{x_1-0}^{z_2} \delta(z_1-x_n) \, dz_1 \cdots dz_I \quad . 1.$$

where: N and I are positive definite integers; $0 < h$; whenever $1 < N$, we have $x_{n+1} - x_n = h$ for every $1 \le n \le N-1$; and δ is Dirac's operator.

INTRODUCTION

The motivation for this paper is the appearance of $D(N,I)$ in the LCM.FORTRAN[1] algorithm which enables macro-engineering analyses of structures exhibiting local or grain-size effects above the micro-scale[2]. In the application of the LCM to a given mesomechanics[3] problem, we can anticipate the integer I to remain fixed at some number $I \sim 100$, while N may take on a dozen (or fewer) values spanning 1 to maxN = 10^6. All of the appearances of N in the LCM algorithm are contained in $D(N,I)$. Therefore, we seek to study computational issues that are contingent to effective computer resources management of equation-1 when applying the LCM to mesomechanics problems.

To sharpen the focus of this study, we call an arithmetic equivalent or "exact" arithmetic restatement of the right-hand-side of equation-1 a "computable formulation of $D(N,I)$" if such a restatement can be said to be <u>readily</u> translatable into FORTRAN statements. In restricting the present discussion to "exact" formulations - rather than including approximations - we, thereby, define the first computational issue of the study. For example, a computable formulation produced by integrating the right-hand-side of equation-1 is

$$D(N,I) = D_1(N,I) \qquad \qquad 2.$$

where:

$$D_1(N,I) = \begin{cases} N \ldots \ldots \ldots I = 1, \quad \text{and} \\[2ex] \dfrac{\sum_{n=1}^{N} n^{I-1}}{(I-1)!} \ldots I \ge 2 . \end{cases} \qquad 3.$$

The second issue concerns various strategies for producing computable formulations of D(N,I). The third addresses value standards for judging a computable formulation to be "good," or for judging one good computable formulation "better" than another. And the fourth issue involves the selection of a sequential, 32-bit word length environment as the basis for extrapolating the results of this study into other computing environments.

In view of anticipated LCM applications in mesomechanics, an a priori judgment of "good" will be assigned to a computable formulation of D(N,I) if arithmetic complexity analyses predict that the cpu required of its implmentation in a sequential environment is independent of the choice of N. We see, for example, that computation of the sum in equation-1 via a "DO" loop statement over the range 1 to N requires cpu that are (linearly) dependent upon the size of N and, therefore, D(N,I) is not a "good" computable formulation of D(N,I). Rationales for judging a given good formulation "better" than another consist of appeals to issues articulating memory requirements, parallelizeabilities, and the range of (N,I) pairs over which the given formulation is tractable (i.e., avoids overflow and is otherwise executable and precise) when implemented in a 32-bit word length environment.

Three computable formulations are produced and ranked according to these issues apparently for the first time here.

The literatures on generalized functions, lattice sums, et al., offer a variety of strategies for otaining computable formulations of D(N,I). However, there seems to be no previous discussion about D(N,I) with respect to the computational issues expressed above. Consideration, therefore, is given to these issues as they relate to an immediately perceivable or "naive" strategy for producing computable formulations of D(N,I). Experiences gained from implementing these formulations lead to development of an informed strategy for producing a better formulation, and experiences gained thereupon lead in turn to an even better formulation.

Production of these "better" formulations together with the rationale given for judging one of them "best" constitute the main points of this paper. The significance and potential impact of the paper derive from the proof given herein that the LCM model of the mesomechanics problem is (1) an O(I)-system of equations and, therefore, (2) a viable alternative to the conventional finite element model (FEM) because the FEM is an O(maxN)-system where O(I) << O(maxN).

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8. CONCLUSIONS AND RECOMMENDATIONS

Conclusions

We have presented closed-form approximations for the frequency of structural systems, both with and without damping. As such, these solutions are valid for all micromechanical models which determine the macromechanical stiffnesses D_{ij}. Contiguity factors, fiber shape factors and misalignment factors only affect D_{ij}, and hence are implicitly included in the results presented. However, for angle-ply symmetric laminates, a parametric study has shown that the frequency is relatively independent of the micromechanical model as long as classical laminate theory applies to the bonding between layers. It is further pointed out that FEM models do not present any inherent problem, for in Chapter 3 a solution was obtained for the general discretized problem.

The viscoelastic analysis presented some greater than anticipated difficulties. Solutions were obtained for generic problems in which the damping could be modelled as a simple damping matrix. For such models, however, the approximate solutions presented appear very promising.

The load correction method has been extended from lattice elements to include solid elements [1]. The case of solid composite elements, however, remains to be demonstrated.

Recommendations

The next phase for the LCM approach should focus on demonstrating the applicability of the composite solid element and the completion of the dynamic analysis using this element to obtain

natural frequencies *for* composites. This could *be* accomplished using either the methods developed by Broome in Chapter 7 or Reiss in Chapter 3.

Subsequent suggested work on sensitivity analysis could focus on various simple viscoelastic models of orthtropic materials - including Maxwell and Kelvin models. Once done, the method of Chapter 4 could be extended to these viscoelastic models.

References

[1] Broome, T.H., "LCM 21", Fortran code residing on the VAX 750 at the School of Engineering, Howard University.

APPENDIX

CONTRACT PRODUCTIVITY

1. Degrees

a. Ph.D. Bo Qian, "On the Application of Green's Functional
      to Distributed Parameter Sensitivity Analysis and Optimum
      Structural Design," July 1990, Howard University.

   Dr. Qian is currently an associate professor of mechanical
   engineering at the University of the Di3strict of Columbia,
   Washington, D.C.

2. Articles, Supported by the Contract

a. Bo Qian and R. Reiss, "The Fundamental Frequency of
   Symmetric Laminates Determined by Eigensensitivity
   Analysis," OPTI/89, in Computer Aided Optimum Design
   of Structures: Recent Advances, Eds., C.A. Brebbia
   and S. Hernandez, Springer-Verlag, Southhampton, UK,
   June, 1989, pp. 299-310.

b. R. Reiss and B. Qian, "Eigenvalues of Self-Adjoint
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d. R. Reiss, O. Barton, L. Thigpen, W. Aung and B. Qian,
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   Conference, Florence, Italy, Sept. 1991.

e. T.H. Broome, "Computational Issues in the Application of
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